Prof. Dr. Emmanuel Kowalski

Exercise Sheet 6

1. Let $d \geq 1$ be an integer and let $\xi = (\xi_1, \dots, \xi_d) \in (\mathbb{R}/\mathbb{Z})^d$ be given. Let T be the closure of the set $\{n\xi | n \in \mathbb{Z}\}$.

a) Prove that T is a closed subgroup of $(\mathbb{R}/\mathbb{Z})^d$.

b) If $1, \xi_1, \dots, \xi_d$ are $\mathbb{Q}$-linearly independent (viewed as real numbers), then

$$T = (\mathbb{R}/\mathbb{Z})^d.$$

2. Let $(X_1, \dots, X_N)$ be complex valued symmetric random variables, i.e., for all $\varepsilon_i = \pm 1$, $1 \leq i \leq N$, $(X_1, \dots, X_N)$ is distributed like $(\varepsilon_1 X_1, \dots, \varepsilon_N X_N)$. Let

$$S_k = X_1 + \dots X_k, \ 1 \leq k \leq N.$$

Let $\varepsilon > 0$, $T = \inf\{k \leq N \mid |S_k| > \varepsilon\}$.

a) Prove that

$$\mathbb{P}(|S_N| > \varepsilon) = \sum_{k=1}^{N} \mathbb{P}(|S_N| > \varepsilon \text{ and } T = k).$$

b) Show that

$$\mathbb{P}(|S_N| > \varepsilon) = \sum_{k=1}^{N} \mathbb{P}(|S_k - R_k| > \varepsilon \text{ and } T = k)$$

where $S_N = S_k + R_k$.

c) Prove that

$$\mathbb{P}\Big(\max_{k \leq N} |S_k| > \varepsilon\Big) \leq 2\mathbb{P}\big(|S_N| > \varepsilon\big).$$

d) Show that if $(X_n)_{n \geq 1}$ is a sequence of symmetric random variables with

$$\sum_n \mathbb{V}(X_n) < +\infty$$

and

$$\mathbb{E}(X_n \overline{X_m}) = 0$$

for all $n \neq m$, then $\sum_{n \geq 1} X_n$ converges almost surely.

e) Give a proof of Proposition 1 seen in the lecture, using d) instead of Kolmogorov's theorem.

Submission: Monday, 2nd November 2015 during the exercise class.