Exercise Sheet 7

1. Let G be a compact abelian group and let $y\colon H \longrightarrow G$ be a continuous homomorphism. Then

$$\overline{\text{Image}(y)} = G$$

if and only if for all characters $\chi \neq 1$ of G, there exists $h \in H$ such that

$$\chi(y(h)) \neq 1.$$

2. (Maximal Inequality) Let $U\colon L^1_\mu \longrightarrow L^1_\mu$ be a positive linear operator with $\|U\| \leq 1$. For $f \in L^1_\mu$ a real-valued function, define the functions

$$f_n = \sum_{k=0}^{n-1} U^k f$$

for $n \geq 0$ (i.e., $f_0 = 0$, $f_1 = f$, $f_2 = f + Uf$), and $F_N = \max\{f_n \mid 0 \leq n \leq N\}$ (all these functions are defined pointwise). Prove that

$$\int_{\{x \mid F_N(x)>0\}} f d\mu \geq 0$$

for all $N \geq 1$.

3. Consider the measure-preserving system $(X, \mathcal{B}, \mu, T)$ on a probability space and g a real-valued function in $\mathscr{L}^1_\mu$. Define

$$E_\alpha = \left\{ x \in X \,\middle|\, \sup_{n\geq 1} \frac{1}{n} \sum_{i=0}^{n-1} g(T^i x) > \alpha \right\}$$

for any $\alpha \in \mathbb{R}$. Show that

$$\alpha\mu(E_\alpha) \leq \int_{E_\alpha} g d\mu \leq \|g\|_1.$$

4. Recall the following version of the mean ergodic theorem:

Theorem (Mean Ergodic Theorem)**.** *Let* $(X, \mathcal{B}, \mu, T)$ *be a measure-preserving system. Then for any function* $f \in L^1_\mu$ *the ergodic averages*

$$A_N(f) = \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n$$

converge in L^1_μ *to a* T*-invariant function* $f' \in L^1_\mu$*.*

The goal of this exercise is to prove the following theorem:
Theorem (Birkhoff)**.** *Let* $(X, \mathcal{B}, \mu, T)$ *be a measure-preserving system. If* $f \in \mathscr{L}^1_\mu$*, then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} f \circ T^j$$

converges almost everywhere and in L^1_μ *to a* T*-invariant function* $f^* \in \mathscr{L}^1_\mu$*, and*

$$\int f^* d\mu = \int f d\mu.$$

a) Show that for $f_0 \in \mathscr{L}^\infty$,

$$A_N\big(A_M(f_0)\big) = A_N(f_0) + O_M\left(\frac{\|f_0\|_\infty}{N}\right).$$

b) Prove Birkhoff's theorem for $f \in \mathscr{L}^\infty$.
Hint: Use the mean ergodic theorem to prove convergence in L^1_μ and use the maximal ergodic theorem together with part a) to show convergence almost everywhere.

c) Prove Birkhoff's theorem as stated.

Submission: Monday, 9th November 2015 during the exercise class.