Exercise Sheet 8

1. Show that if
$$f(s_0) = \sum_{n\geq 1} \frac{a_n}{n^{s_0}}$$
converges, then for Re$(s) > \sigma_0$
$$|f(s)| \ll (1+|t|)^c$$
uniform for $\sigma_0 < \sigma_1 \leq \text{Re}(s) \leq \sigma_2$, where c is some constant which may depend on s_0, σ_1 and σ_2.

2. Show that
$$|\zeta(\sigma+it)| \ll (1+|t|)$$
for $2 \geq \sigma \geq \frac{1}{2}$ and $|t| \geq 1$. Can you explicitly compute an implied constant?

3. Let
$$\sum_{n=1}^{\infty} |a_n| < +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} |a_n n^{it}| < +\infty.$$

a) Show that
$$\int_{-T}^{T} \Big| \sum_{n=1}^{\infty} a_n n^{it} \Big|^2 dt \ll T^2 \int_0^{\infty} \Big| \sum_{y \leq n \leq y e^{\frac{1}{T}}} a_n \Big|^2 \frac{dy}{y}.$$

b) Use a) to prove that for $T \geq 1$,
$$\int_{-T}^{T} \Big| \sum_{n=1}^{\infty} a_n n^{-it} \Big|^2 dt \ll \sum_{n=1}^{\infty} |a_n|^2 (n+T).$$

c) Use b) to show that
$$\int_{-T}^{T} |\zeta(\sigma+it)|^2 dt \ll \begin{cases} T & \text{if } \sigma > \frac{1}{2}, \\ T\log T & \text{if } \sigma = \frac{1}{2}. \end{cases}$$

Submission: Monday, 16th November 2015 during the exercise class.