Exercise Sheet 9

1. Suppose that $\frac{1}{2} < \sigma < 1$ is fixed.

a) Let $(a_0, a_1, \ldots, a_m) \in \mathbb{C}^{m+1}$ with $a_0 \neq 0$. Show that there exists some vector $(b_0, b_1, \ldots, b_m) \in \mathbb{C}^{m+1}$ for which

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) = \sum_{k=0}^{m} \frac{a_k}{k!} s^k + O\big(s^{m+1}\big)$$

holds.
Hint: Use induction over m.

b) Let g be a nowhere vanishing holomorphic function and let $\varepsilon > 0$. Show that there exists $\tau \in \mathbb{R}$ and $r > 0$ such that

$$\max_{|s| \le r} |\zeta(s + \sigma + i\tau) - g(s)| < \frac{\varepsilon r^k}{k!}$$

for all $k \in \{0, \ldots, n-1\}$.
Hint: Use Voronin's universality theorem.

c) Let $\varepsilon > 0$. Show that there exists $\tau \in \mathbb{R}$ such that

$$|\zeta^{(k)}(\sigma + i\tau) - g^{(k)}(0)| < \varepsilon$$

for all $k \in \{0, \ldots, n-1\}$.
Hint: Use Cauchy's integral formula and part b).

d) Consider the map $\varphi_\sigma \colon \mathbb{R} \longmapsto \mathbb{C}^n$ given by

$$\varphi_\sigma(t) = \big(\zeta(\sigma + it), \zeta^{(1)}(\sigma + it), \ldots, \zeta^{(n-1)}(\sigma + it)\big).$$

Show that $\varphi_\sigma(\mathbb{R})$ is dense in $\mathbb{C}^n$, i.e., that $\overline{\varphi_\sigma(\mathbb{R})} = \mathbb{C}^n$.

2. Let $F_0, \ldots, F_N \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ be continuous functions, not all identically vanishing. Show that there exists some $s \in \mathbb{C}$ such that

$$\sum_{k=0}^{N} s^k F_k\big(\zeta(s), \zeta^{(1)}(s), \ldots, \zeta^{(n-1)}(s)\big) \neq 0.$$

In particular, ζ does not satisfy any algebraic differential equation.

Submission: Monday, 23th November 2015 during the exercise class.