Solution 1

1. a) We first show that for all integers $n \geq 1$,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{1}$$

This is clearly true for $n = 1$. Therefore, we assume that $n > 1$ and write

$$n = p_1^{a_1} \cdots p_k^{a_k}.$$

In the sum $\sum_{d|n} \mu(d)$ the only nonzero terms come from those divisors of n which are products of distinct primes ($d = 1$ is the empty product). Thus

$$\sum_{d|n} \mu(d) = \sum_{I \subset \{p_1,\ldots,p_k\}} \mu\Big(\prod_{p \in I} p\Big) = \sum_{n=0}^{k} \sum_{\substack{I \subset \{p_1,\ldots,p_k\} \\ |I|=n}} \mu\Big(\prod_{p \in I} p\Big)$$
$$= \sum_{n=0}^{k} \sum_{\substack{I \subset \{p_1,\ldots,p_k\} \\ |I|=n}} (-1)^n = \sum_{n=0}^{k} \binom{k}{n} (-1)^n = (1-1)^k = 0$$

This completes the proof of (1).

Denote by m^2 the square part of n, i.e., let m be the biggest number such that $m^2 \mid n$. Then

$$\sum_{d^2|n} \mu(d) = \sum_{d^2|m^2} \mu(d) = \sum_{d|m} \mu(d) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

But $m = 1$ if and only if n is squarefree.

b) By part c), we have

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d} = \sum_{d|n} n\mu\Big(\frac{n}{d}\Big).$$

By exercise 2, part b), this is equivalent to

$$n = \sum_{d|n} \varphi(d).$$

Bitte wenden!

c) We have

$$\varphi(n) = \sum_{\substack{1\le k\le n\\(n,k)=1}} 1 = \sum_{k=1}^{n} \sum_{d|(n,k)} \mu(d) = \sum_{k=1}^{n} \sum_{\substack{d|n\\d|k}} \mu(d) = \sum_{d|n} \mu(d) \sum_{1\le \ell \le \frac{n}{d}} 1$$
$$= \sum_{d|n} \mu(d) \frac{n}{d}.$$

2. a) Note that the definition of $f * g$ can be rewritten as

$$(f * g)(n) = \sum_{ab=n} f(a)f(b).$$

Hence $f * g = g * f$ is obvious. Furthermore

$$\big((f * g) * k\big)(n) = \sum_{dc=n} (f * g)(d)k(c) = \sum_{dc=n} \sum_{ab=d} f(a)g(b)k(c)$$
$$= \sum_{abc=n} f(a)g(b)k(c) = \sum_{ad=n} f(a) \sum_{bc=d} g(b)k(c)$$
$$= \sum_{ad=n} f(a)(g * k)(d) = \big(f * (g * k)\big)(n).$$

b) We define the unit function $u\colon \mathbb{N} \longrightarrow \mathbb{C}$ by $u(n) = 1$ for all n and the identity function $I\colon \mathbb{N} \longrightarrow \mathbb{C}$ by

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Note that I is the identity element of the Dirichlet multiplication, i.e. for any arithmetic function f we have

$$f * I = f.$$

Furthermore note that

$$(\mu * u)(n) = \sum_{d|n} \mu(d) = I(n).$$

Hence u is the inverse of μ with respect to the Dirichlet multiplication. Thus

$$f * \mu = g$$

implies

$$f = f * \mu * u = g * u$$

and vice versa.

Siehe nächstes Blatt!

3. a) Let

$$A = \bigcup_{i=1}^{n} A_i$$

and denote by $\mathbb{1}_{A_i} \colon A \longrightarrow \{0,1\}$ the characteristic function of A_i. Clearly, for every $x \in A$,

$$\prod_{i=1}^{n} \big(1 - \mathbb{1}_{A_i}(x)\big) = 0.$$

Therefore,

$$\begin{aligned} 0 = \prod_{i=1}^{n} \big(1 - \mathbb{1}_{A_i}(x)\big) &= \sum_{I \subset \{1,\dots,n\}} \prod_{i \in I} \big(- \mathbb{1}_{A_i}(x)\big) \\ &= 1 + \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|} \prod_{i \in I} \mathbb{1}_{A_i}(x) \\ &= 1 + \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|} \mathbb{1}_{\bigcap_{i \in I} A_i}(x). \end{aligned}$$

which can be written as

$$1 = \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \mathbb{1}_{\bigcap_{i \in I} A_i}(x) \tag{2}$$

Finally, we obtain

$$\begin{aligned} \left| \bigcup_{i=1}^{n} A_i \right| = |A| = \sum_{x \in A} 1 &= \sum_{x \in A} \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \mathbb{1}_{\bigcap_{i \in I} A_i}(x) \\ &= \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \sum_{x \in A} \mathbb{1}_{\bigcap_{i \in I} A_i}(x) \\ &= \sum_{\substack{I \subset \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|. \end{aligned}$$

b) Note that (2) also holds for measurable sets $A_1, \dots, A_n$. Hence

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{i=1}^n A_i\Big) &= \mathbb{P}(A) = \int_A 1 d\mu \\
&= \int_A \sum_{\substack{I\subset\{1,\dots,n\}\\ I\neq\emptyset}} (-1)^{|I|-1} \mathbb{1}_{\bigcap_{i\in I} A_i}(x) d\mu \\
&= \sum_{\substack{I\subset\{1,\dots,n\}\\ I\neq\emptyset}} (-1)^{|I|-1} \int_A \mathbb{1}_{\bigcap_{i\in I} A_i}(x) d\mu \\
&= \sum_{\substack{I\subset\{1,\dots,n\}\\ I\neq\emptyset}} (-1)^{|I|-1} \int_{\bigcap_{i\in I} A_i} 1 d\mu \\
&= \sum_{\substack{I\subset\{1,\dots,n\}\\ I\neq\emptyset}} (-1)^{|I|-1} \mathbb{P}\Big(\bigcap_{i\in I} A_i\Big).
\end{aligned}$$

4. a) There are many proofs of this identity. The classical proofs normally use some form of the functional equation of the Riemann zeta function along with some contour integration in the complex plane (see for example [1, Theorem 12.17]). This kind of proofs are normally a bit lengthy, but they use standard methods and are straightforward.

There are also many short and/or elementary proofs. The disadvantage of these proofs is, that the methods used are often ad hoc and do not give much insight. See for example [2] and the references there for a list of different proofs.

b) This follows directly from part a).

5. We have to show that

$$\zeta(s) \sum_{m\geq 1} \frac{\mu(m)}{m^s} = 1.$$

Using the definition of ζ, we have

$$\zeta(s) \sum_{m\geq 1} \frac{\mu(m)}{m^s} = \sum_{n\geq 1} \frac{1}{n^s} \sum_{m\geq 1} \frac{\mu(m)}{m^s}$$

and since both series are absolutely convergent, this is

$$\begin{aligned}
&= \sum_{k\geq 1} \frac{1}{k^s} \sum_{mn=k} \mu(m) \\
&= \sum_{k\geq 1} \frac{1}{k^s} \sum_{m|k} \mu(m) = 1,
\end{aligned}$$

Siehe nächstes Blatt!

since

$$\sum_{m|k} \mu(m) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer, 1976

[2] E. de Amo, M. Díaz Carrillo, J. Fernández-Sánchez, Another proof of Euler's formula for $\zeta(2k)$, Proc. Amer. Math. Soc. 139 (2011) 1441-1444