Solution 10

1. a) We have

$$\begin{aligned}\theta(x)-1 = 2\sum_{n\geq 1} e^{-\pi n^2 x} &\leq 2\sum_{n\geq 1} e^{-\pi n x}\\ &\leq 2e^{-pix} + 2\int_1^\infty e^{-\pi n x}dn \leq 2e^{-\pi x} + 2\frac{e^{-\pi x}}{\pi x}\\ &\ll e^{-\pi x}\end{aligned}$$

for all $x \geq 1$.

b) This is a direct consequence of the Poisson summation formula.

c) Recall the definition of the gamma function

$$\Gamma(s) = \int_0^{+\infty} e^{-x}x^{s-1}dx$$

for Re$(s) > 1$. For all $n \geq 1$, we have

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)n^{-s} = \int_0^{+\infty} \pi^{-s/2}n^{-s}e^{-x}x^{\frac{s}{2}-1}dx$$

which is by the change of variable $x = \pi n^2 y$,

$$\begin{aligned}&= \int_0^{+\infty} \pi^{-s/2}n^{-s}e^{-\pi n^2 y}(\pi n^2 y)^{\frac{s}{2}-1}\pi n^2 dy\\ &= \int_0^{+\infty} e^{-\pi n^2 y}y^{\frac{s}{2}-1}dy.\end{aligned}$$

Summing over all $n \geq 1$, we get

$$\begin{aligned}\Lambda(s) = \pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) &= \sum_{n\geq 1}\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)n^{-s}\\ &= \sum_{n\geq 1}\int_0^{+\infty} e^{-\pi n^2 y}y^{\frac{s}{2}-1}dy = \int_0^{+\infty}\sum_{n\geq 1} e^{-\pi n^2 y}y^{\frac{s}{2}-1}dy\\ &= \frac{1}{2}\int_0^{+\infty}\big(\theta(y)-1\big)y^{\frac{s}{2}-1}dy,\end{aligned}$$

where

$$\theta(y) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}.$$

d) By a) we have that

$$\theta(x) - 1 \ll e^{-\pi x} \tag{1}$$

for all $x \geq 1$. We write

$$\int_0^{+\infty} \big(\theta(y) - 1\big) y^{\frac{s}{2}-1} dy = \int_0^1 \big(\theta(y) - 1\big) y^{\frac{s}{2}-1} dy + \int_1^{+\infty} \big(\theta(y) - 1\big) y^{\frac{s}{2}-1} dy.$$

Note that by (1), the second integral is an entire function. Also note that by the Poisson summation formula

$$\theta(x) = \frac{1}{\sqrt{x}} \theta(x^{-1}).$$

Hence

$$\int_0^1 \big(\theta(y) - 1\big) y^{\frac{s}{2}-1} dy = \int_0^1 \frac{1}{\sqrt{y}} \big(\theta(y^{-1}) - 1\big) y^{\frac{s}{2}-1} dy + \int_0^1 \left(\frac{1}{\sqrt{y}} - 1\right) y^{\frac{s}{2}-1} dy$$

which is by setting $x = \frac{1}{y}$,

$$\begin{aligned} &= \int_1^{\infty} \sqrt{x} \big(\theta(x) - 1\big) x^{1-\frac{s}{2}} \frac{dx}{x^2} + \int_0^1 \left(\frac{1}{\sqrt{y}} - 1\right) y^{\frac{s}{2}-1} dy \\ &= \int_1^{\infty} \big(\theta(x) - 1\big) x^{\frac{1-s}{2}} \frac{dx}{x} + \int_0^1 y^{\frac{s-1}{2}} \frac{dy}{y} - \int_0^1 y^{\frac{s}{2}} \frac{dy}{y} \\ &= \int_1^{\infty} \big(\theta(x) - 1\big) x^{\frac{1-s}{2}} \frac{dx}{x} + \frac{2}{s-1} - \frac{2}{s}. \end{aligned}$$

Because

$$\frac{2}{s-1} - \frac{2}{s} = \frac{2}{s(s-1)}$$

we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^{+\infty} \big(\theta(y) - 1\big) y^{\frac{s}{2}-1} dy \\ &\qquad = \frac{-1}{s(1-s)} + \frac{1}{2} \int_1^{+\infty} \big(\theta(y) - 1\big) y^{\frac{s}{2}-1} dy + \frac{1}{2} \int_1^{\infty} \big(\theta(x) - 1\big) x^{\frac{1-s}{2}} \frac{dx}{x} \end{aligned}$$

which is what we had to show.

e) By c) and d)

$$\Lambda(s) = \frac{-1}{s(1-s)} + \frac{1}{2} \int_1^{+\infty} \big(\theta(y) - 1\big) y^{\frac{s}{2}-1} dy + \frac{1}{2} \int_1^{\infty} \big(\theta(x) - 1\big) x^{\frac{1-s}{2}} \frac{dx}{x}. \tag{2}$$

The integral expressions on the right hand side of the equation are entire functions by a). Since $\frac{-1}{s(1-s)}$ has simple poles at 0 and 1 we get that Λ is a meromorphic function on $\mathbb{C}$ with simple poles at 0 and 1. It is also obvious from the representation (2), that Λ satisfies the functional equation

$$\Lambda(s) = \Lambda(1-s).$$

Siehe nächstes Blatt!

2. a) We have

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2\cos^2\theta - 1 = 2(1+\cos\theta)^2 \geq 0$$

with equality only when $\cos\theta = -1$ or, equivalently, $\theta = (2k+1)\pi$ for some $k \in \mathbb{Z}$.

b) From the Euler product we have that for Re$(s) > 1$,

$$\zeta(s) = \prod_p \left(\frac{1}{1-p^{-s}}\right).$$

Taking logarithm, we get that

$$\log\zeta(s) = -\sum_p \log(1-p^{-s}) = \sum_p \sum_{n\geq 1} \frac{p^{-ns}}{n}$$

and hence

$$\text{Re}\big(\log\zeta(s)\big) = \sum_p \sum_{n\geq 1} \frac{p^{-n\sigma}\cos(t\log p^m)}{n}. \tag{3}$$

Using (3) three times, we get

$$\begin{aligned}\text{Re}\log D(\sigma) &= \sum_p \sum_{n\geq 1} \frac{p^{-n\sigma}\big(3\cos(0) + 4\cos(t\log p^m) + \cos(2t\log p^m)\big)}{n} \\ &\geq 0.\end{aligned}$$

by part a) with $\theta = t\log p^m$. We conclude that

$$\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geq 1 \tag{4}$$

for $\sigma > 1$. Now, for $t = 0$ the conclusion is obvious. For $t \neq 0$, we have as $\sigma \to 1$ that $\zeta \sim \frac{1}{\sigma-1}$ and $\zeta(\sigma+it) \in O(\sigma-1)$. Hence $\zeta(\sigma)^3\zeta(\sigma+it)^4 \in O(\sigma-1)$. By (4) this would imply that ζ has a pole at $1+2it$ which contradicts the fact that ζ is analytic at $1+2it$. Thus the conclusion follows.

c) Follows trivially from b).

3. We will make use of Perron's formula. If

$$q(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\text{Re}(s) > \sigma$ then

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} q(\xi)\frac{x^\xi}{\xi}d\xi$$
$$= \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} q(\xi)\frac{x^\xi}{\xi}d\xi \tag{5}$$

with the convention that, in the above sum, if x is an integer, the last term in the sum appears with coefficient $\frac{1}{2}$. A proof of this statement can be found, for instance, in [1, Theorem 11.18]. Since

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = \sum_p \frac{(\log p)p^{-s}}{1-p^{-s}}$$
$$= \sum_p (\log p)\sum_{n\geq 1} p^{-ns}$$
$$= \sum_{n\geq 1}\Lambda(n)n^{-s},$$

the conclusion is a direct application of (5).

4. We make use of the Residue Theorem. If γ is a positively oriented simple closed curve and f is a meromorphic function with poles a_k inside γ we have that

$$\frac{1}{2\pi i}\oint_\gamma f(z)dz = \sum_k \text{Res}(f, a_k)$$

where $\text{Res}(f, a_k)$ denotes the residue of f at the point a_k. For our purposes, we will choose

$$f(s) = -\left(\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}.$$

Pick $M, T > 0$ and $c > 1$. Let γ_1 be the rectangular curve starting at the point $-M-iT$ going through $c-iT$, $c+iT$, $-M+iT$ and back to $-M-iT$ in this order. We apply the residue theorem to the chosen function f. We have to discuss its poles inside γ.

i) ζ has a simple pole at 1 and hence f has a pole at $s = 1$ with residue x.

ii) The denominator s gives a pole for f at $s = 0$ with residue

$$-\frac{\zeta'(0)}{\zeta(0)} = -\frac{-\frac{1}{2}\log 2\pi}{-\frac{1}{2}} = -\log 2\pi.$$

iii) The zeroes of ζ in the half-plane $s < 0$ inside γ, namely, $s = -2k, k \in \mathbb{Z}_{>0}, 2k \leq M$ translate into poles for f inside γ with residues $-\frac{x^{2k}}{2k}$.

Siehe nächstes Blatt!

iv) The zeroes ϱ in the critical strip and inside γ give poles for f with residues $-\frac{x^\varrho}{\varrho}$. Because they are inside γ they satisfy $|\mathrm{Im}(\varrho)| < T$ We get

$$\frac{1}{2\pi i}\oint_\gamma f(z)dz = x + \sum_{k=1}^{\lfloor \frac{M}{2} \rfloor} \frac{x^{-2k}}{2k} - \log(2\pi) - \sum_{\substack{\varrho \text{ non-trivial} \\ \text{zero of } \zeta, \\ |\mathrm{Im}(\varrho)|<T}} \frac{x^\varrho}{\varrho} \tag{6}$$

We split the integral on the left hand side of (6) into four subintegrals

$$I_1 = \frac{1}{2\pi i}\int_{-M-iT}^{C-iT} f(z)dz,$$

$$I_2 = \frac{1}{2\pi i}\int_{C-iT}^{C+iT} f(z)dz,$$

$$I_3 = \frac{1}{2\pi i}\int_{C+iT}^{-M+iT} f(z)dz,$$

$$I_4 = \frac{1}{2\pi i}\int_{-M+iT}^{-M-iT} f(z)dz.$$

We need to estimate I_1, I_3 and I_4. I_2 will be estimated using Exercise 3. The estimations for I_1 and I_4 are completely equivalent. We will now use the following result

$$\frac{\zeta'(s)}{\zeta(s)} \in O(\log|s|) \tag{7}$$

for large $|s|$. We will not prove this result. For a complete solution and how to chose T such that γ doesn't pass through any zero of ζ see [3]. Note that (7) implies, after some manipulations, that I_4 vanishes as $M \to \infty$. Also (7) implies that

$$I_1, I_3 \in O\left(\frac{x^c \log T}{T}\right) \tag{8}$$

and (8) in turn implies that I_1, I_3 vanish as $T \to \infty$. The discussion above implies that if we let $M \to \infty$ and $T \to \infty$ in (6) for I_1, I_3 and I_4 and use Exercise 3 for I_2 we get the desired conclusion.

5. See for example [2, Corollary 3.3.5].

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer, 1976

[2] E. Kowalski, Un course de théorie analytique des nombres.

[3] Yum-Tong Siu, Lecture notes: *Explicit Formula for Logarithmic Derivative of Riemann Zeta Function*, 2005, http://www.math.harvard.edu/archive/213b_spring_05/explicit_formula_log _derivative_zeta_function.pdf