Solution 11

1. a) If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ with $p_i \le X$ and $a_1 + \ldots + \alpha_r = k$, then we have

$$\frac{k!}{\alpha_1! \cdots \alpha_r!} = \sum_{\substack{2 \le p_1, \ldots, p_k \le X \\ p_1 \cdots p_k = n}} 1.$$

Hence

$$\begin{aligned}
\sum_{m \ne n} a_k(m) a_\ell(n) &\le \sum_{m,n} a_k(m) a_\ell(n) \\
&= \sum_{m,n} \sum_{\substack{2 \le p_1, \ldots, p_k \le X \\ p_1 \cdots p_k = n}} 1 \sum_{\substack{2 \le p_1, \ldots, p_\ell \le X \\ p_1 \cdots p_\ell = m}} 1 \\
&= \sum_{n} \sum_{\substack{2 \le p_1, \ldots, p_k \le X \\ p_1 \cdots p_k = n}} 1 \sum_{m} \sum_{\substack{2 \le p_1, \ldots, p_\ell \le X \\ p_1 \cdots p_\ell = m}} 1 \\
&= \sum_{2 \le p_1, \ldots, p_k \le X} 1 \sum_{2 \le p_1, \ldots, p_\ell \le X} 1 \\
&\le X^{k+\ell}.
\end{aligned}$$

b) First, note that

$$\mathcal{P}_0(s)^k = \left(\sum_{p \le X} \frac{1}{p^s} \right)^k = \sum_{p_1, \ldots, p_k \le X} \frac{1}{(p_1 \cdots p_k)^s} = \sum_{n \ge 1} \frac{1}{n^s} \sum_{\substack{p_1, \ldots, p_k \le X \\ p_1 \cdots p_k = n}} 1 = \sum_{n \ge 1} \frac{a_k(n)}{n^s}.$$

Therefore

$$\begin{aligned}
\int_T^{2T} \mathcal{P}_0(\sigma_0 + it)^k \mathcal{P}_0(\sigma_0 - it)^\ell dt &= \sum_{n \ge 1} \sum_{m \ge 1} \frac{a_k(m) a_\ell(n)}{(mn)^{\sigma_0}} \int_T^{2T} \frac{1}{m^{it} n^{-it}} dt \\
&= T \sum_{n \ge 1} \frac{a_k(n) a_\ell(n)}{n^{2\sigma_0}} \\
&\quad + \sum_{m \ne n} \frac{a_k(m) a_\ell(n)}{(mn)^{\sigma_0}} \int_T^{2T} \frac{1}{m^{it} n^{-it}} dt.
\end{aligned}$$

Since

$$\int_T^{2T} \frac{1}{m^{it} n^{-it}} dt \ll \frac{1}{|\log(m/n)|}$$

we get the desired result.

Bitte wenden!

c) First we consider the case $k \neq \ell$. From the lecture, we know that

$$\frac{1}{|\log(m/n)|} \ll \sqrt{mn}.$$

Also note that for $k \neq \ell$, $a_k(n)a_\ell(n) = 0$. Hence by b),

$$\int_T^{2T} \mathcal{P}_0(\sigma_0 + it)^k \mathcal{P}_0(\sigma_0 - it)^\ell dt \ll \sqrt{mn} \sum_{m \neq n} \frac{a_k(m)a_\ell(n)}{(mn)^{\sigma_0}}$$
$$\ll \sum_{m \neq n} a_k(m)a_\ell(n)$$

which is by part a)

$$\ll X^{k+\ell} \leq T.$$

The case $k = \ell$ is a bit more complicated, see [1, Lemma 1] for a solution.

2. a) For $n = p_1 \cdots p_r$ squarefree, we have

$$\sum_{\substack{h,k \\ [h,k]=n}} \mu\big((h,k)\big) = \sum_{m=0}^{r} (-1)^m |\{(h,k) \mid \omega((h,k)) = m,\ [h,k] = n\}|$$
$$= \sum_{m=0}^{r} \binom{r}{m} 2^{r-m}(-1)^m$$
$$= (2-1)^r = 1.$$

b) We compute

$$\sum_{\substack{h,k \\ p|hk \Rightarrow p \leq Y}} \frac{\mu(h)\mu(k)}{(hk)^s}(h,k)^s = \sum_{\substack{h,k \\ p|hk \Rightarrow p \leq Y}} \frac{\mu\big([h,k]\big)\mu\big((h,k)\big)}{[h,k]^s}$$
$$= \sum_{\substack{h,k \\ [h,k] \text{ squarefree} \\ p|[h,k] \Rightarrow p \leq Y}} \frac{\mu\big([h,k]\big)\mu\big((h,k)\big)}{[h,k]^s}$$
$$= \sum_{\substack{n \text{ squarefree} \\ p|n \Rightarrow p \leq Y}} \frac{\mu(n)}{n^s} \sum_{\substack{h,k \\ [h,k]=n}} \mu\big((h,k)\big)$$

which is by part a)

$$= \sum_{\substack{n \text{ squarefree} \\ p|n \Rightarrow p \leq Y}} \frac{\mu(n)}{n^s} = \prod_{p \leq Y} \Big(1 - \frac{1}{p^s}\Big).$$

Siehe nächstes Blatt!

3. See Proposition 2.4 on page 19 in [**?**].

4. See Proposition 2.13 on page 25 in [**?**].

5. See Proposition 2.15 on page 26 in [**?**].

References

[1] E. Kowalski, Lecture notes: *Exponential sums over finite fields, I: elementary methods*, http://www.math.ethz.ch/~kowalski/exp-sums.pdf.

[2] M. Radziwiłłand K. Soundararajan, *Selberg's central limit theorem for* $\log|\zeta(\frac{1}{2}+it)|$, 2015, arXiv:1509.06827