Solution 12

1. You can find a detailed exposition of a), b) and d) in [1] on pages 21, 22, 23 and 24. Part c) is a trivial consequence of the fact that if x achieves all values in $\mathbb{F}_p^\times$ then bx also achieves all values in $\mathbb{F}_p^\times$.

2. a) Set $\alpha = \frac{k\pi}{q}$. We have that

$$\begin{aligned}\left|1 - e\left(\frac{k}{q}\right)\right| &= \sqrt{(1-\cos 2\alpha)^2 + \sin^2 2\alpha} \\ &= \sqrt{2 - 2\cos 2\alpha} = 2|\sin\alpha| = 2\sin\alpha. \qquad (1)\end{aligned}$$

We want to prove that for $0 \leq x \leq \frac{\pi}{2}$ the inequality

$$\sin x \geq \frac{x}{2} \qquad (2)$$

holds. For this, consider the function

$$f(x) = \sin x - \frac{x}{2}$$

Its derivative is

$$f'(x) = \cos x - \frac{1}{2}$$

We get that on the interval $[0, \frac{\pi}{3}]$, f is increasing and on the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$, f is decreasing. Hence the minimum of f on the interval $[0, \frac{\pi}{2}]$ is attained on the extremes of the interval. Since $f(0) = 0$ and $f(\frac{\pi}{2}) = 1 - \frac{\pi}{4} > 0$ it immediately follows that (2) holds. Using (2) in (1) we get that

$$\left|1 - e\left(\frac{k}{q}\right)\right| = 2\sin\frac{k\pi}{q} \geq \pi\frac{k}{q}$$

b) We have that

$$\sum_{k=1}^{q-1}\left|\sum_{n\leq X} e\left(\frac{kn}{q}\right)\right| = \sum_{k=1}^{q-1}\left|\frac{1 - e\left(\frac{k(\lfloor X\rfloor+1)}{q}\right)}{1 - e\left(\frac{k}{q}\right)}\right| \leq 2\sum_{k=1}^{q-1}\left|\frac{1}{1-e\left(\frac{k}{q}\right)}\right| \qquad (3)$$

By using part a) in (3) we obtain

$$\sum_{k=1}^{q-1}\left|\sum_{n\leq X} e\left(\frac{kn}{q}\right)\right| \leq 2\sum_{k=1}^{q-1}\left|\frac{1}{1-e\left(\frac{k}{q}\right)}\right| \leq 2\sum_{k=1}^{q-1}\left|\frac{1}{\pi\frac{k}{q}}\right| = \frac{2}{\pi}q\sum_{k=1}^{q-1}\frac{1}{k} \ll q\log q.$$

c) Let $\hat{\chi}$ be the normalized Fourier transform of χ i.e.

$$\hat{\chi}(k) = \frac{1}{\sqrt{q}} \sum_{n=1}^{q-1} \chi(n) e\left(-\frac{kn}{q}\right)$$

Applying the inverse Fourier transform gives us

$$\chi(n) = \frac{1}{\sqrt{q}} \sum_{n=1}^{q-1} \hat{\chi}(k) e\left(\frac{kn}{q}\right)$$

It is a standard calculation to check that $|\hat{\chi}(k)| = 1$. See [1], page 19, Proposition 2.4 for details. We then get that

$$\sum_{n \leq X} \chi(n) = \frac{1}{\sqrt{q}} \sum_{n \leq X} \sum_{k=1}^{q-1} \hat{\chi}(k) e\left(\frac{kn}{q}\right) = \frac{1}{\sqrt{q}} \sum_{k=1}^{q-1} \hat{\chi}(k) \left(\sum_{n \leq X} e\left(\frac{kn}{q}\right)\right)$$

Using $|\hat{\chi}(k)| = 1$, the triangle inequality and part b), we conclude that

$$\begin{aligned} \textstyle\sum_{n \leq X} \chi(n) &\leq \left|\textstyle\sum_{n \leq X} \chi(n)\right| = \left|\tfrac{1}{\sqrt{q}} \textstyle\sum_{k=1}^{q-1} \hat{\chi}(k) \left(\sum_{n \leq X} e\left(\frac{kn}{q}\right)\right)\right| \\ &\leq \tfrac{1}{\sqrt{q}} \textstyle\sum_{k=1}^{q-1} \left|\sum_{n \leq X} e\left(\frac{kn}{q}\right)\right| \ll \tfrac{1}{\sqrt{q}} q \log q = \sqrt{q} \log q. \end{aligned}$$

3. If k is odd then the function x^k is odd so the integral is trivially 0. If k is even, let $k = 2n$ with $n \in \mathbb{Z}_{\geq 0}$. We want to evaluate

$$I_n = \frac{1}{\pi} \int_{-2}^{2} x^{2n} \sqrt{1 - \frac{x^2}{4}} dx.$$

Consider the change of variable $x = 2 \sin \theta$. When x varies between -2 and 2, θ varies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Taking into account that the $\cos$ function is nonnegative on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and that $dx = 2\cos\theta d\theta$, we get

$$\begin{aligned} I_n &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n} \sin^{2n} \theta \sqrt{1 - \frac{4 \sin^2 \theta}{4}} 2 \cos \theta d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n+1} \sin^{2n} \theta \cos^2 \theta d\theta \\ &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n+1} \sin^{2n} \theta (1 - \sin^2 \theta) d\theta \end{aligned}$$

Set

$$b_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n} \sin^{2n} \theta d\theta$$

Then,

$$I_n = 2b_n - \frac{1}{2} b_{n+1}. \tag{4}$$

Siehe nächstes Blatt!

Evaluating b_{n+1} gives

$$\begin{aligned} b_{n+1} &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n+2}\sin^{2n+2}\theta d\theta = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n+1}\sin^{2n}\theta(2\sin^2\theta)d\theta \\ &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n+1}\sin^{2n}\theta(1-\cos 2\theta)d\theta \\ &= 2\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n}\sin^{2n+2}\theta d\theta - \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n}\sin^{2n}\theta(2\cos 2\theta)d\theta \end{aligned}$$

which is by applying integration by parts

$$\begin{aligned} &= 2b_n - \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n}\sin^{2n}\theta(\sin 2\theta)' d\theta \\ &= 2b_n + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n}2n\sin^{2n-1}\theta\cos\theta\sin 2\theta d\theta \\ &= 2b_n + 2n\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^{2n}\sin^{2n-1}\theta\cos\theta(2\sin\theta\cos\theta)d\theta = 2b_n - 2nI_n. \end{aligned}$$

Combining this with (4) gives

$$b_{n+1} = 2b_n + 4nb_n - nb_{n+1}$$

and hence

$$(n+1)b_{n+1} = (4n+2)b_n.$$

Since $b_0 = 1$ we get

$$\begin{aligned} b_n &= \prod_{\ell=1}^{n}\frac{4\ell-2}{k} = \frac{\prod_{\ell=1}^{n}(4\ell-2)}{\ell!} = \frac{2^n\prod_{\ell=1}^{n}(2\ell-1)}{n!} = \frac{2^n n!\prod_{\ell=1}^{n}(2\ell-1)}{n!n!} \\ &= \frac{(2n)!}{n!n!} = \binom{2n}{n} \end{aligned}$$

Hence,

$$I_n = 2\binom{2n}{n} - \frac{1}{2}\binom{2n+2}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

and we are done.

Literatur

[1] E. Kowalski, Lecture notes: *Exponential sums over finite fields, I: elementary methods*, http://www.math.ethz.ch/~kowalski/exp-sums.pdf.