Solution 2

1. This is a very standard result. For a detailed proof see for example [1, Theorem 1.10].

2. This result is also standard. A proof can be found for example in [1, Theorem 11.7]. Note that there is also a slightly more general version, see [1, Theorem 11.6].

3. By summation by parts, we have

$$\sum_{p\leq x}\frac{1}{p}=\frac{\pi(x)}{x}-\frac{\pi(2)}{2}+\int_2^x\frac{\pi(t)}{t^2}dt$$

which is by the prime number theorem

$$\begin{aligned}&=\int_2^x\frac{1}{t\log t}dt+O\bigg(\int_1^x\frac{1}{t\log^2 t}dt\bigg)+O(1)\\&=\big[\log\log t\big]_2^x+O(1)\\&=\log\log x+O(1).\end{aligned}$$

Note that we have used the following version of the prime number theorem:

$$\pi(x)=\frac{x}{\log x}+O\bigg(\frac{x}{\log^2 x}\bigg).$$

4. a) We compute

$$\begin{aligned}|\{1\leq n\leq N\mid n\text{ is squarefree}\}|&=\sum_{\substack{n\leq N\\ n\text{ squarefree}}}1=\sum_{n\leq N}\sum_{d^2\mid n}\mu(d)\\&=\sum_{d\leq\sqrt{N}}\mu(d)\sum_{\substack{n\leq N\\ d^2\mid n}}1\\&=\sum_{d\leq\sqrt{N}}\mu(d)\bigg(\frac{N}{d^2}+O(1)\bigg)\\&=N\sum_{d\leq\sqrt{N}}\frac{\mu(d)}{d^2}+O\big(\sqrt{N}\big).\end{aligned}$$

Hence, we get that

$$\frac{1}{N}|\{1 \leq n \leq N \mid n \text{ is squarefree}\}| = \sum_{d \leq \sqrt{N}} \frac{\mu(d)}{d^2} + O\big(N^{-\frac{1}{2}}\big)$$

and by taking the limit $N \to \infty$, this becomes

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

b) We compute

$$\begin{aligned}
&|\{(n_1, n_2) \in \Omega_N \times \Omega_N \mid (n_1, n_2) = 1\}| \\
&\qquad = \sum_{\substack{1 \leq n_1, n_2 \leq N \\ (n_1, n_2) = 1}} 1 = \sum_{1 \leq n_1, n_2 \leq N} \sum_{d \mid (n_1, n_2)} \mu(d) = \sum_{1 \leq n_1, n_2 \leq N} \sum_{\substack{d \mid n_1 \\ d \mid n_2}} \mu(d) \\
&\qquad = \sum_{1 \leq d \leq N} \mu(d) \sum_{1 \leq k, \ell \leq \frac{N}{d}} 1 = \sum_{1 \leq d \leq N} \mu(d) \left(\frac{N}{d} + O(1) \right)^2 \\
&\qquad = \sum_{1 \leq d \leq N} \mu(d) \left(\frac{N^2}{d^2} + O\left(\frac{N}{d} \right) \right) = N^2 \sum_{1 \leq d \leq N} \frac{\mu(d)}{d^2} + O(N \log N).
\end{aligned}$$

Therefore

$$\lim_{N \to \infty} \frac{1}{N^2} |\{(n_1, n_2) \in \Omega_N \times \Omega_N \mid (n_1, n_2) = 1\}| = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

References

[1] Tom M. Apostol, Introduction to Analytic Number Theory, Springer, 1976