Solution 4

1. Recall that

$$\sum_{d^k \mid P(n)} \mu(d) = \begin{cases} 1 & \text{if } P(n) \text{ is } k\text{-free}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \frac{1}{N}|\{1 \leq n \leq N \mid P(n) \text{ is } k\text{-free}\}| &= \frac{1}{N}\sum_{n \leq N}\sum_{d^k \mid P(n)} \mu(d) \\ &= \sum_{d \leq cN^{\frac{m}{k}}} \mu(d) \frac{1}{N} \sum_{\substack{n \leq N \\ d^k \mid P(n)}} 1 \\ &= \sum_{d \leq cN^{\frac{m}{k}}} \mu(d) \left(\nu_{d^k}(0) + O_P\left(\frac{1}{N}\right)\right) \\ &= \sum_{d \leq cN^{\frac{m}{k}}} \mu(d)\nu_{d^k}(0) + O_P\big(N^{\frac{m}{k}-1}\big). \end{aligned}$$

2. a) First, note that $\mu(d) \neq 0$ if and only if d is squarefree. Hence it suffices to consider squarefree d. We need to compute the number of solutions of

$$P(x) \equiv 0 \mod d^k.$$

By the Chinese Remainder Theorem and the fact that d is squarefree, it is enough to determine the number of solutions of

$$P(x) \equiv 0 \mod p^k$$

for every $p \mid d$. The total number of solutions is then the product of these number of solutions. Since P is of degree m, it is clear that

$$P(x) \equiv 0 \mod p$$

has at most m solutions. If $p \nmid \Delta$, P has only single roots modulo p and hence $P'(x) \not\equiv 0 \mod p$ for every root x of P. Therefore, by Hensel's Lemma, there exist a unique lift for every root modulo p to a root modulo p^k. Hence

$$P(x) \equiv 0 \mod p^k$$

has at most m solutions. On the other hand, if $p \mid \Delta$, $P'(x) \equiv 0 \mod p$ for some roots x of P modulo p. Thus, we can not say anything about how the roots lift and hence in the worst case

$$P(x) \equiv 0 \mod p^k$$

has mp^{k-1} roots. Thus we get

$$|\{x \in \mathbb{Z}/d^k\mathbb{Z} \mid P(x) \equiv 0 \mod d^k\}| \leq \prod_{\substack{p|d \\ p\nmid\Delta}} m \prod_{\substack{p|d \\ p|\Delta}} mp^{k-1} = m^{\omega(d)}(d,\Delta)^{k-1}.$$

b) By a) we have that

$$\sum_{d=1}^{N} |\mu(d)\nu_{d^k}(0)| \leq \sum_{d\leq N} \frac{m^{\omega(d)}}{d^k}(d,\Delta)^{k-1} \leq \Delta^{k-1} \sum_{d\leq N} \frac{m^{\omega(d)}}{d^k},$$

which converges by the integral test. Hence

$$\sum_{d=1}^{\infty} \mu(d)\nu_{d^k}(0) = \prod_p \left(1 - \nu_{p^k}(0)\right)$$

by the Euler product formula, the product being absolutely convergent.

3. We know already that

$$x^2 + 1 \equiv 0 \mod p$$

has 2 solutions if $p \equiv 1 \mod 4$ and 0 solutions if $p \equiv 3 \mod 4$. The discriminant of $X^2 + 1$ is -4 and hence only $p = 2$ divides the discriminant. Thus, again by Hensel's lemma, we have that

$$x^2 + 1 \equiv 0 \mod p^3$$

has 2 solutions if $p \equiv 1 \mod 4$ and 0 solutions if $p \equiv 3 \mod 4$. It remains to consider $p = 2$. A direct computation show, that $x^2+1 \equiv 0 \mod 2^3$ has no solutions.

Hence we get by exercise 1

$$\lim_{N\to\infty} \frac{1}{N}|\{1 \leq n \leq N \mid n^2 + 1 \text{ is 3-free}\}| = \sum_{d=1}^{\infty} \mu(d)\nu_{d^3}(0)$$

which is by exercise 2

$$\begin{aligned} &= \prod_p \left(1 - \nu_{p^3}(0)\right) \\ &= \prod_{p\equiv 1 \mod 4} \left(1 - \frac{2}{p^3}\right) \\ &\approx 0.9825146. \end{aligned}$$