# Solution 5

**1.** Let $K$ be a compact set in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > \sigma_0\}$. Then there exists $\sigma_1 > \sigma_0$ such that for every $s \in K$, $\mathrm{Re}(s) \geq \sigma_1$ and hence $\sigma_0 - \mathrm{Re}(s) \leq \sigma_0 - \sigma_1 < 0$.

Fix $\varepsilon > 0$. Since

$$\sum_{n\geq 1} \frac{a_n}{n^{\sigma_0}}$$

converges, there exists $n_0 \geq 1$ such that for all $m, m' \geq n_0$,

$$\left| \sum_{n=m}^{m'} \frac{a_n}{n^{\sigma_0}} \right| < \varepsilon. \tag{1}$$

We can choose $n_0$ such that also $n_0^{\sigma_0-\sigma_1} < \varepsilon'$. Hence

$$\begin{aligned}
\sum_{n=m}^{m'} a_n n^{-s} &= \sum_{n=m}^{m'} a_n n^{-s+\sigma_0-\sigma_0} \\
&= \sum_{n=m}^{m'} a_n n^{-\sigma_0} n^{\sigma_0 - s} \\
&= (m')^{\sigma_0-s} \sum_{n=m}^{m'} a_n n^{-\sigma_0} - m^{\sigma_0-s} \sum_{n=m}^{m'} a_n n^{-\sigma_0} + \int_m^{m'} t^{\sigma_0-s-1} \sum_{n=m}^{t} a_n n^{-\sigma_0} dt
\end{aligned}$$

By the triangle inequality and (1), we get

$$\begin{aligned}
\left| \sum_{n=m}^{m'} a_n n^{-s} \right| &\leq \varepsilon (m')^{\sigma_0-\mathrm{Re}(s)} + \varepsilon m^{\sigma_0-\mathrm{Re}(s)} + \varepsilon \int_m^{m'} t^{\sigma_0-\mathrm{Re}(s)-1} dt \\
&= \varepsilon (m')^{\sigma_0-\mathrm{Re}(s)} + \varepsilon m^{\sigma_0-\mathrm{Re}(s)} + \varepsilon m^{\sigma_0-\mathrm{Re}(s)} - \varepsilon (m')^{\sigma_0-\mathrm{Re}(s)} \\
&\leq \varepsilon n_0^{\sigma_0-\mathrm{Re}(s)} + \varepsilon n_0^{\sigma_0-\mathrm{Re}(s)} + \varepsilon n_0^{\sigma_0-\mathrm{Re}(s)} \\
&\leq 3\varepsilon\varepsilon'.
\end{aligned}$$

This gives uniform convergence on $K$.

**2.** The proof is analogous to the one for Exercise 1, except that $\varepsilon$ is replaced by $B$ and $\sigma_0$ by $0$.

**3.** If $f$ is identically zero, we are done. So, assume that $f$ is not identically zero on $D$ and assume by contradiction that $f$ has some zeros in the interior $D$. Since $f$ is holomorphic, these zeros are isolated. So if $z_0$ is one of this zeros, we find $\varrho > 0$ such that $f(z) \neq 0$ for all $z$ with $0 < |z - z_0| \leq \varrho$. Hence there exists $m > 0$ such that $|f(z)| \geq m$ for all $z$ with $|z - z_0| = \varrho$. Having fixed $\varrho$ and $m$, we can choose $n_0$ so large that

$$|f_n(z) - f(z)| < m$$

for all $n > n_0$ on the circle $|z - z_0| = \varrho$. Since $f_n(z) = f(z) + \big(f_n(z) - f(z)\big)$, it follows from Rouché's theorem that, for $n > n_0$, $f_n(z)$ has the same number of zeros in the circle as $f(z)$; hence $f_n(z)$ has a zero in $D$, contradicting the assumption.

**4.** See Theorem B.2.7 in the lecture notes.