Solution 6

1. a) Recall that if G is a topological group and H a subgroup of G, then $\overline{H}$ is also a subgroup of G.

Proof. Since G is a topological group, the map

$$f\colon G \times G \longrightarrow G,\ (x,y) \longmapsto xy^{-1}$$

is continuous. Hence $f^{-1}\big(\overline{H}\big)$ is closed in $G \times G$. Furthermore, by definition of f and since H is a subgroup, we have

$$H \times H \subset f^{-1}\big(\overline{H}\big).$$

Thus $\overline{H} \times \overline{H} \subset \overline{H \times H} \subset f^{-1}\big(\overline{H}\big)$. Consequently

$$f\big(\overline{H} \times \overline{H}\big) \subset \overline{H}$$

which shows that $\overline{H}$ is a subgroup of G. □

Since $\{n\xi | n \in \mathbb{Z}\}$ is obviously a subgroup of the topological group $(\mathbb{R}/\mathbb{Z})^d$, it follows immediately that T is a subgroup of $(\mathbb{R}/\mathbb{Z})^d$.
Alternatively, we can also directly show that T is closed under addition and taking inverses using the fact that these maps are continuous. To do so, pick $y = (y_1, \dots, y_d) \in T$. First we show that $-y \in T$. Since T is the closure of $\{n\xi | n \in \mathbb{Z}\}$ we have that for every $\varepsilon > 0$ there exist $m \in \mathbb{Z}$ such that

$$m\xi \in (y_1 - \varepsilon, y_1 + \varepsilon) \times \ldots \times (y_d - \varepsilon, y_d + \varepsilon).$$

But then

$$-m\xi \in (-y_1 - \varepsilon, -y_1 + \varepsilon) \times \ldots \times (-y_d - \varepsilon, -y_d + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, we get $-y \in T$.
Similarly, if $a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d) \in T$ then for every $\varepsilon > 0$ there exist $m, n \in \mathbb{Z}$ such that

$$m\xi \in (a_1 - \varepsilon/2, a_1 + \varepsilon/2) \times \ldots \times (a_d - \varepsilon/2, a_d + \varepsilon/2)$$
$$n\xi \in (b_1 - \varepsilon/2, b_1 + \varepsilon/2) \times \ldots \times (b_d - \varepsilon/2, b_d + \varepsilon/2).$$

But then

$$(p+q)\xi \in (a_1 + b_1 - \varepsilon, a_1 + b_1 + \varepsilon) \times \ldots \times (a_d + b_d - \varepsilon, a_d + b_d + \varepsilon).$$

Again, since $\varepsilon > 0$ was chosen arbitrarily, we obtain $a + b \in T$. Thus T is a closed subgroup of $(\mathbb{R}/\mathbb{Z})^d$.

Bitte wenden!

b) Define $T_{\geq 0}$ as the closure of $\{n\xi | n \in \mathbb{Z}_{\geq 0}\}$. Since $T_{\geq 0} \subseteq T$, it is enough to show $T_{\geq 0} = (\mathbb{R}/\mathbb{Z})^d$. For this we make use of Weyl's criterion. Define the sequence $(a_n)_{n\geq 0}$ by $a_n = n\xi$. Pick $l = (l_1, \ldots, l_d) \in \mathbb{Z}^d$, $l \neq 0$ arbitrarly. Notice that because the ξ are all irrational and $\mathbb{Q}$-linearly independent, $e(\xi \cdot l)$ cannot be equal to 1. We have that

$$\frac{1}{N+1}\sum_{n=0}^{N} e(a_n \cdot l) = \frac{1}{N+1}\sum_{n=0}^{N} e(\xi \cdot l)^n = \frac{1}{N+1}\frac{1-e(\xi\cdot l)^{N+1}}{1-e(\xi\cdot l)}.$$

But then by the triangle inequality we get that $|1-e(\xi\cdot l)^{N+1}| \leq 1+1 = 2$. Hence

$$\left|\frac{1}{N+1}\sum_{n=0}^{N} e(a_n\cdot l)\right| \leq \frac{1}{N+1}\frac{2}{|1-e(\xi\cdot l)|}.$$

Hence we get that as $N \to \infty$

$$\frac{1}{N+1}\sum_{n=0}^{N} e(a_n\cdot l) \to 0$$

which by Weyl's criterion implies that $(a_n)_{n\geq 0}$ is equidistributed in $(\mathbb{R}/\mathbb{Z})^d$. But this trivially implies that $T_{\geq 0}$ is dense in $(\mathbb{R}/\mathbb{Z})^d$ and since $T_{\geq 0}$ is closed, we obtain that $T = T_{\geq 0} = (\mathbb{R}/\mathbb{Z})^d$.

2. a) Consider the events

$$A = \{|S_N| > \varepsilon\}, \qquad B = \{1 \leq T \leq N\} \text{ and } \qquad B_k = \{T = k\}, \tag{1}$$

for $k \in \{1, \ldots, N\}$. Notice that $B = \sqcup_k B_k$. Moreover it is easy to see that if $|S_N| > \varepsilon$ then $1 \leq \inf\{k \leq N| \, |S_k| > \varepsilon\} \leq N$. Hence $A \subset B$. Since B is a disjoint union of the B_k's we get

$$\begin{aligned}\mathbb{P}(|S_N| > \varepsilon) &= \mathbb{P}(A) = \mathbb{P}(A\cap B) = \mathbb{P}(A \cap (\sqcup_k B_k)) \\ &= \mathbb{P}(\sqcup_k (A\cap B_k)) = \sum_{k=1}^{N} \mathbb{P}(A\cap B_k) \\ &= \sum_{k=1}^{N} \mathbb{P}(|S_N| > \varepsilon \text{ and } T = k).\end{aligned}$$

b) Define the random variables $Y_i = X_i$ for $1 \leq i \leq k$ and $Y_i = -X_i$ for $k < i \leq N$. Set

$$\tilde{S}_m = Y_1 + \ldots + Y_m$$

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and $\tilde{T} = \inf\{k \leq N | \, |\tilde{S}_k| > \varepsilon\}$. It is easy to see that for $i \leq k$ we have that $\tilde{S}_i = S_i$ and so $\{T = k\} = \{\tilde{T} = k\}$. Using the symmetry hypothesis we get that

$$\mathbb{P}(|S_N| > \varepsilon, T = k) = \mathbb{P}(|\tilde{S}_N| > \varepsilon, \tilde{T} = k) = \mathbb{P}(|S_k - R_k| > \varepsilon, T = k)$$

for $1 \leq k \leq N$. Summing over k gives us the desired conclusion.

c) By adding the equalities of a) and b) we get that

$$\sum_{k=1}^{N} \big(\mathbb{P}(|S_N| > \varepsilon \text{ and } T = k) + \mathbb{P}(|S_k - R_k| > \varepsilon \text{ and } T = k)\big) = 2\mathbb{P}(|S_N| > \varepsilon).$$

Define

$$C_k = \{|S_k - R_k| > \varepsilon\}$$

and recall the definitions made in (1). Hence we can rewrite the previous equality as

$$\sum_{k=1}^{N} (\mathbb{P}(A \cap B_k) + \mathbb{P}(C_k \cap B_k)) = 2\mathbb{P}(A).$$

Notice that the event $\big\{ \max_{k \leq N} |S_k| > \varepsilon \big\}$ is the same as B because if $\max_{k \leq N} |S_k| > \varepsilon$ then $1 \leq T \leq i$ for any i for which $|S_i| > \varepsilon$. Conversely, if $1 \leq T \leq N$ then $|S_T| > \varepsilon$ and so $\max_{k \leq N} |S_k| \geq |S_T| > \varepsilon$. So we need to show that

$$\mathbb{P}(B) \leq 2\mathbb{P}(A).$$

Notice that if $|S_k + R_k| \leq \varepsilon$ and $|S_k - R_k| \leq \varepsilon$ then by the triangle inequality $2|S_k| \leq |S_k + R_k| + |S_k - R_k| \leq 2\varepsilon$ and so $|S_k| \leq \varepsilon$. But this means that if $|S_k| > \varepsilon$ then either $|S_k + R_k| > \varepsilon$ or $|S_k - R_k| > \varepsilon$. We just showed that $B_k \subset (A \cup C_k)$. Hence

$$B_k \subset (A \cap B_k) \cup (C_k \cap B_k)$$

and so

$$\mathbb{P}(B_k) \leq \mathbb{P}(A \cap B_k) + \mathbb{P}(C_k \cap B_k).$$

Adding over k gives us

$$\mathbb{P}(B) = \sum_{k=1}^{N} \mathbb{P}(B_k) \leq \sum_{k=1}^{N} (\mathbb{P}(A \cap B_k) + \mathbb{P}(C_k \cap B_k)) = 2\mathbb{P}(A).$$

Thus we are done.

d) The proof is very similar to the one seen in the lecture, except that we have to replace the arguments which make use of the independence of the X_i's. We want to show that the sequence of partial sums

$$S_N = X_1 + \ldots + X_N$$

is almost surely a Cauchy sequence. For this purpose denote,

$$Y_{N,M} = \sup_{1 \le k \le M} |S_{N+k} - S_N|$$

for $N, M \ge 1$. For fixed N, $Y_{N,M}$ is an increasing sequence of random variables; we denote by $Y_N = \sup_{k \ge 1} |S_{N+k} - S_N|$ its limit. Because of the estimate

$$|S_{N+k} - S_{N+l}| \le |S_{N+k} - S_N| + |S_{N+l} - S_N| \le 2Y_N$$

for $N \ge 1$ and $k, l \ge 1$, we have

$$\begin{aligned}\{(S_N)_{N\ge 1} \text{ is not Cauchy}\} &= \bigcup_{i\ge 1} \bigcap_{N\ge 1} \bigcup_{k\ge 1} \bigcup_{l \ge 1} \{|S_{N+k} - S_{N+l}| > 2^{-i}\} \\ &\subset \bigcup_{i \ge 1} \bigcap_{N\ge 1} \{Y_N > 2^{-i-1}\}.\end{aligned}$$

The last part is due to the previous inequality. It is therefore sufficient to prove that

$$\mathbb{P}(\bigcap_{N\ge 1} \{Y_N > 2^{-i-1}\}) = 0$$

for each $i \ge 1$, or equivalently, it is sufficient to prove that for $\varepsilon > 0$ we have

$$\lim_{N\to\infty} \mathbb{P}(Y_N > \varepsilon) = 0.$$

So far we followed the line of proof in the notes. What follows does not use the independence assumption. Notice that by part c) we have that

$$\mathbb{P}(Y_{N,M} > \varepsilon) \le 2\mathbb{P}(|S_{N+M} - S_N| > \varepsilon).$$

Applying Markov's inequality we get

$$\mathbb{P}(Y_{N,M} > \varepsilon) \le 2\frac{\mathbb{E}(|S_{N+M} - S_N|)}{\varepsilon}.$$

Let us show that $\mathbb{E}(|S_{N+M} - S_N|)$ is a Cauchy sequence in M for every $N \ge 0$. Because

$$\sum_n \mathbb{V}(X_n) < \infty$$

we get that

$$\sum_{n=1}^{N} \mathbb{E}(|X_n|^2)$$

is Cauchy. Also

$$\mathbb{E}(|S_{N+M} - S_N|)^2 \le \mathbb{E}(|S_{N+M} - S_N|^2) = \mathbb{E}(|X_{N+1}|^2) + \ldots + \mathbb{E}(|X_{N+M}|^2).$$

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The last part is because $\mathbb{E}(X_n\overline{X_m}) = 0$ for $n \neq m$. Hence $\mathbb{E}(|S_{N+M} - S_N|)$ is indeed Cauchy. Now since

$$\mathbb{P}(Y_{N,M} > \varepsilon) \leq 2\frac{\mathbb{E}(|S_{N+M} - S_N|)}{\varepsilon}$$

we obtain by taking $M \to \infty$ that

$$\mathbb{P}(Y_N > \varepsilon) \leq 2\frac{\mathbb{E}\left(\left|\sum_{k\geq 1} X_{N+k}\right|\right)}{\varepsilon}.$$

Using the fact that $\mathbb{E}(|S_N|)$ is Cauchy (which is the above case for $N = 0$ and M replaced with N), we easily get that

$$\lim_{N\to\infty} \mathbb{P}(Y_N > \varepsilon) = 0.$$