Solution 7

1. For the only if part, assume that there exists a character $\chi \neq 1$ of G, such that for every $h \in H$,

$$\chi(y(h)) = 1.$$

Hence $\mathrm{Image}(y) \subset \ker(\chi)$. Since χ is non-trivial, the kernel of χ is a proper closed subgroup of G and thus

$$\overline{\mathrm{Image}(y)} \subset \ker(\chi) \subsetneq G.$$

For the if part, assume that $\overline{\mathrm{Image}(y)} \neq G$. Then $\overline{\mathrm{Image}(y)}$ is a proper closed subgroup of G. Hence $G/\overline{\mathrm{Image}(y)}$ is a non-trivial compact group and hence there exists a non-trivial character

$$\tilde{\chi} \colon G/\overline{\mathrm{Image}(y)} \longrightarrow \mathbb{C}^{\times}.$$

If we denote by π the projection map

$$\pi \colon G \longrightarrow G/\overline{\mathrm{Image}(y)}$$

then $\chi = \tilde{\chi} \circ \pi$ is a non-trivial character of G. But by definition of χ, $\overline{\mathrm{Image}(y)} \subset \ker(\chi)$, which completes the proof.

2. For a solution, see [1, Proposition 2.26].

3. This is Theorem 2.24 on page 38 in [1].

4. For a solution, see the second proof of [1, Theorem 2.30], which is on page 46 and 47.

References

[1] M. Einsiedler, T. Ward, Ergodic Theory with a view towards Number Theory, Springer, 2011