Solution 8

1. Set

$$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$$

and

$$F_k(s) = \sum_{n=1}^{k} \frac{a_n}{n^s}$$

We proceed similarly to Exercise 1 in Sheet 5. There exists $K > 0$ such that $|F_N(\sigma_0) - F_M(\sigma_0)| < 1$ for all $N, M \geq K$. We have that for $N > K$

$$|F_N(s)| \leq |F_K(s)| + |F_N(s) - F_K(s)|.$$

We have to estimate the two terms on the RHS. For the first one, notice that

$$|F_K(s)| \leq \sum_{n=1}^{K} \frac{|a_n|}{n^{\sigma}} \leq \sum_{n=1}^{K} \frac{|a_n|}{n^{\sigma_1}}.$$

Denote the last sum by A which is a non-negative constant depending only on σ_0 and σ_1.

For the second term we use Abel summation as follows

$$\begin{aligned}\sum_{n=K+1}^{N} a_n n^{-s} &= \sum_{n=K+1}^{N} a_n n^{-\sigma_0} n^{\sigma_0 - s} \\ &= N^{\sigma_0 - s} \sum_{n=K+1}^{N} a_n n^{-\sigma_0} - K^{\sigma_0 - s} \sum_{n=K+1}^{N} a_n n^{-\sigma_0} + \\ &+ (\sigma_0 - s) \int_{K+1}^{N} x^{\sigma_0 - s - 1} \sum_{n=K+1}^{x} a_n n^{-\sigma_0} dx\end{aligned}$$

In the above we will use the triangle inequality. We will also use the fact that for $\sigma > \sigma_0, n^{\sigma_0 - \sigma} \leq 1$ and that all partial sums appearing are bounded above by 1 because of the choice of K. We get

$$\begin{aligned}|F_N(s)-F_K(s)| &\le 1\cdot 1+1\cdot 1+|\sigma_0-s|\int_{K+1}^{N} x^{\sigma_0-\sigma-1}\cdot 1dx\\ &=2+\frac{|\sigma_0-s|}{\sigma_0-\sigma}(\sigma_0-\sigma)\int_{K+1}^{N} x^{\sigma_0-\sigma-1}dx\\ &=2+\frac{|\sigma_0-s|}{\sigma-\sigma_0}((K+1)^{\sigma_0-\sigma}-N^{\sigma_0-\sigma})\\ &\le 2+2\frac{|\sigma_0-s|}{\sigma-\sigma_0}\le 2+2\frac{|\sigma_0-s|}{\sigma_1-\sigma_0}\end{aligned}$$

Observe that

$$|\sigma_0-s|^2=(\sigma-\sigma_0)^2+t^2\le(\sigma_2-\sigma_0)^2+t^2<B^2(1+|t|)^2$$

for some positive constant B that depends only on σ_0 and σ_2. Thus

$$|F_N(s)|<A+2+\frac{2B}{\sigma_1-\sigma_0}(1+|t|)$$

This immediately implies that there exists a positive constant C depending only on $\sigma_0,\sigma_1,\sigma_2$ such that

$$|F_N(s)|<C(1+|t|)$$

Taking $N\to\infty$ gives us the desired result.

2. By summation by parts, we get for $\sigma>0$

$$\zeta(s)=\frac{s}{s-1}-s\int_1^\infty\frac{\{x\}}{x^{s+1}}dx,$$

where $\{x\}$ denotes the fractional part of x. Using the triangle inequality and $\{x\}\le 1$ we get

$$|\zeta(s)|\le\left|\frac{s}{s-1}\right|+|s|\int_1^\infty\frac{1}{x^{\sigma+1}}dx=\left|1+\frac{1}{s-1}\right|+\frac{|s|}{\sigma}\le 1+\left|\frac{1}{s-1}\right|+\frac{|s|}{\sigma}$$

Now

$$\left|\frac{1}{s-1}\right|^2=\left|\frac{1}{\sigma-1+it}\right|^2=\frac{1}{(\sigma-1)^2+t^2}\le\frac{1}{t^2}\le 1\le t^2.$$

Hence

$$\left|\frac{1}{s-1}\right|\le|t|.$$

Similarly

$$\left(\frac{|s|}{\sigma}\right)^2=\frac{\sigma^2+t^2}{\sigma^2}=1+\frac{1}{\sigma^2}t^2\le 1+4t^2<4(1+|t|)^2$$

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and so

$$\frac{|s|}{\sigma} < 2(1+|t|).$$

Adding up we get

$$|\zeta(\sigma+it)| < 3(1+|t|).$$

3. a) We rewrite

$$\int_{-T}^{T} |\sum_n a_n n^{-it}|^2 dt = \int_{-T}^{T} \Big|\sum a_n e\Big(-\frac{t}{2\pi}\log n\Big)\Big|^2 = \int_{-T}^{T} |S(t)|^2 dt$$

where

$$S(t) = \sum_{n=1}^{\infty} c(\nu_n) e(\nu_n t)$$

and $\nu_n = -\frac{\log n}{2\pi}$ is a sequence in $\mathbb{R}$. Let $\delta > 0$ be fixed. We have

$$c_\delta(x) := \frac{1}{\delta} \sum_{\substack{n\geq 1 \\ |\nu_n - x| \leq \frac{\delta}{2}}} c(\nu_n) = \sum_{n\geq 1} c(\nu_n) f_\delta(x-\nu_n)$$

where

$$f_\delta(x) = \begin{cases} \delta^{-1} & \text{if } |x| \leq \frac{1}{2}\delta, \\ 0 & \text{if } |x| > \frac{1}{2}\delta. \end{cases}$$

By computing the Fourier Transform of $c_\delta(x)$ we obtain

$$\begin{aligned} \hat{c}_\delta(x) &= \int_{-\infty}^{\infty} c_\delta(y) e(xy) dy = \int_{-\infty}^{\infty} e(xy) \sum_{n\geq 1} c(\nu_n) f_\delta(y-\nu_n) dy \\ &= \sum_{n\geq 1} c(\nu_n) \int_{-\infty}^{\infty} e\big(x(y-\nu_n)\big) e(x\nu_n) f_\delta(y-\nu_n) dy \\ &= \sum_{n\geq 1} c(\nu_n) e(x\nu_n) \hat{f}_\delta(x) = S(x)\hat{f}_\delta(x). \end{aligned}$$

By Plancherel's formula, we therefore get

$$\int_{\mathbb{R}} |c_\delta(x)|^2 dx = \int_{\mathbb{R}} |\hat{c}_\delta(x)|^2 dx = \int_{\mathbb{R}} |S(t)\hat{f}_\delta(t)|^2 dt.$$

Also note that

$$\hat{f}_\delta(t) = \int_{-\infty}^{\infty} f_\delta(x) e(xt) dx = \frac{1}{\delta} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} e(xt) dx = \frac{\sin(\delta\pi t)}{\delta\pi t}.$$

Bitte wenden!

b) By the Cauchy-Schwarz inequality

$$\Big|\sum_{n=y}^{ye^{\frac{1}{T}}} a_n\Big|^2 \le \sum_{n=y}^{ye^{\frac{1}{T}}}|a_n|^2 \sum_{m=y}^{ye^{\frac{1}{T}}} 1 \le \sum_{n=y}^{ye^{\frac{1}{T}}}|a_n|^2\big(y(e^{\frac{1}{T}}-1)+1\big). \tag{1}$$

By the previous exercise,

$$\int_{-T}^{T}\Big|\sum_{n=1}^{\infty} a_n n^{-it}\Big|^2 dt \ll T^2\int_0^\infty \Big|\sum_{n=y}^{ye^{\frac{1}{T}}} a_n\Big|\frac{dy}{y}$$

which is by (2)

$$\begin{aligned}
&\ll T^2\int_0^\infty \sum_{n=y}^{ye^{\frac{1}{T}}}|a_n|^2\big(y(e^{\frac{1}{T}}-1)+1\big)\frac{dy}{y}\\
&= T^2\int_0^\infty \sum_{n=1}^{\infty}\delta_{y\le n\le ye^{\frac{1}{T}}}|a_n|^2\big(y(e^{\frac{1}{T}}-1)+1\big)\frac{dy}{y}\\
&= T^2\sum_{n=1}^{\infty}|a_n|^2\int_0^\infty \delta_{y\le n\le ye^{\frac{1}{T}}}\big(y(e^{\frac{1}{T}}-1)+1\big)\frac{dy}{y}\\
&= T^2\sum_{n=1}^{\infty}|a_n|^2\int_{ne^{-\frac{1}{T}}}^{n}\big(y(e^{\frac{1}{T}}-1)+1\big)\frac{dy}{y}\\
&= T^2\sum_{n=1}^{\infty}|a_n|^2\Big[e^{\frac{1}{T}}y-y+\log y\Big]_{ne^{-\frac{1}{T}}}^{n}\\
&= T^2\sum_{n=1}^{\infty}|a_n|^2\Big(n(e^{\frac{1}{T}}-2+e^{-\frac{1}{T}})+\frac{1}{T}\Big)\\
&= \sum_{n=1}^{\infty}|a_n|^2\big(n(e^{\frac{1}{T}}-1)(1-e^{-\frac{1}{T}})T^2+T\big).
\end{aligned}$$

Since for $T \ge 1$,

$$(e^{\frac{1}{T}}-1) \ll \frac{1}{T} \quad \text{and} \quad (1-e^{-\frac{1}{T}}) \ll \frac{1}{T},$$

we finally get that

$$\int_{-T}^{T}\Big|\sum_{n=1}^{\infty} a_n n^{-it}\Big|^2 dt \ll \sum_{n=1}^{\infty}|a_n|^2(n+T).$$

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c) We prove the statement for $\sigma = \frac{1}{2}$. The case $\sigma > \frac{1}{2}$ can be done similarly. We split the integral into three parts

$$I = \int_{-T}^{T} \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt = I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \int_{-1}^{1} \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt \\ I_2 &= \int_{-T}^{-1} \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt \\ I_3 &= \int_{1}^{T} \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt \end{aligned}$$

Notice that I_1 does not depend on T so we do not need to do any estimation on it. However, we do need to estimate I_2 and I_3. We will estimate I_3. I_2 can then be done similarly. For this purpose we use the following approximation of ζ,

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \tag{2}$$

uniformly for $\sigma \ge \sigma_0 > 0$, $|t| < \frac{2\pi x}{C}$ where C is a given constant with $C > 1$. Now for $t \in [\tilde{T}, 2\tilde{T}]$ we get using (2) that

$$\begin{aligned} \left|\zeta\left(\tfrac{1}{2} + it\right)\right| &= \left| \sum_{n \le \tilde{T}} \frac{1}{n^{\frac{1}{2}+it}} - \frac{\tilde{T}^{1-\frac{1}{2}-it}}{1 - \frac{1}{2} - it} + O(\tilde{T}^{-\frac{1}{2}}) \right| \\ &\le \left| \sum_{n \le \tilde{T}} n^{-\frac{1}{2}} n^{-it} \right| + \frac{\tilde{T}^{\frac{1}{2}}}{\left|\frac{1}{2} - it\right|} + O(\tilde{T}^{-\frac{1}{2}}). \end{aligned} \tag{3}$$

The inequality above is just the triangle inequality. For $t \in [\tilde{T}, 2\tilde{T}]$ we have that $\left|\frac{1}{2} - it\right| > \tilde{T}$. Inserting this in (3) we get

$$\begin{aligned} \left|\zeta\left(\tfrac{1}{2} + it\right)\right| &\ll \left| \sum_{n \le \tilde{T}} n^{-\frac{1}{2}} n^{-it} \right| + \frac{\tilde{T}^{\frac{1}{2}}}{\tilde{T}} + \tilde{T}^{-\frac{1}{2}} \\ &\ll \left| \sum_{n \le \tilde{T}} n^{-\frac{1}{2}} n^{-it} \right| + \tilde{T}^{-\frac{1}{2}}. \end{aligned} \tag{4}$$

For the last sum we use the triangle inequality again and get the following estimate,

$$\left| \sum_{n \le \tilde{T}} n^{-\frac{1}{2}} n^{-it} \right| \le \sum_{n \le \tilde{T}} n^{-\frac{1}{2}} \ll \tilde{T}^{\frac{1}{2}}. \tag{5}$$

Squaring (4) and then using (5) gives us

$$\begin{aligned}\left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 &\ll \left|\sum_{n\le\tilde{T}} n^{-\frac{1}{2}}n^{-it}\right|^2 + 2\left|\sum_{n\le\tilde{T}} n^{-\frac{1}{2}}n^{-it}\right|\tilde{T}^{-\frac{1}{2}} + \tilde{T}^{-1} \\ &\ll \left|\sum_{n\le\tilde{T}} n^{-\frac{1}{2}}n^{-it}\right|^2 + 2\tilde{T}^{\frac{1}{2}}\tilde{T}^{-\frac{1}{2}} + \tilde{T}^{-1} \\ &\ll \left|\sum_{n\le\tilde{T}} n^{-\frac{1}{2}}n^{-it}\right|^2 + 1.\end{aligned} \tag{6}$$

We now use (6) in the following estimation,

$$\begin{aligned}\int_{\tilde{T}}^{2\tilde{T}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt &\ll \int_{\tilde{T}}^{2\tilde{T}} \left|\sum_{n\le\tilde{T}} n^{-\frac{1}{2}}n^{-it}\right|^2 + 1\, dt \\ &\ll \int_{\tilde{T}}^{2\tilde{T}} \left|\sum_{n\le\tilde{T}} n^{-\frac{1}{2}}n^{-it}\right|^2 dt + \tilde{T} \\ &\ll \int_{-2\tilde{T}}^{2\tilde{T}} \left|\sum_{n\le\tilde{T}} n^{-\frac{1}{2}}n^{-it}\right|^2 dt + \tilde{T}.\end{aligned} \tag{7}$$

For the last integral in (7) we use part b) with $a_n = n^{-\frac{1}{2}}$. This gives us

$$\begin{aligned}\int_{\tilde{T}}^{2\tilde{T}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt &\ll \sum_{n\le\tilde{T}} \left|n^{-\frac{1}{2}}\right|^2 (n+2\tilde{T}) + \tilde{T} \\ &\ll \sum_{n\le\tilde{T}} \left(1+\frac{2\tilde{T}}{n}\right) + \tilde{T} \\ &\ll \tilde{T} + 2\tilde{T}\log\tilde{T} + \tilde{T} \ll \tilde{T}\log\tilde{T}.\end{aligned} \tag{8}$$

Now, we split the interval $[1, T]$ into the dyadic intervals $[2^k, 2^{k+1}]$ with $0 \le k \le \log_2 T$. Using (7) on each of these intervals we get

$$\begin{aligned}\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt &\le \sum_{0\le\log_2 T} \int_{2^k}^{2^{k+1}} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \\ &\ll \sum_{0\le\log_2 T} 2^k \log(2^k)\end{aligned} \tag{9}$$

Finally, which using standard summations in the last sum in (8), we deduce that

$$I_3 = \int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt \ll \frac{-T + T\log T}{\log 2} \ll T\log T.$$