Solution 9

1. a) We proceed by induction on m. For $m = 0$ write $a_0 = |a_0|e^{iw}$ with $0 \leq w < 2\pi$. Set $b_0 = \log|a_0| + iw$. It is immediate that $\exp(b_0) = a_0$. Assume now the statement to be true for some $m \geq 0$. We have that

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) = \sum_{k=0}^{m} \frac{a_k}{k!} s^k + O(s^{m+1}).$$

It follows then that there exists $\alpha \in \mathbb{C}$ such that

$$\exp\left(\sum_{k=0}^{m} b_k s^k\right) = \sum_{k=0}^{m} \frac{a_k}{k!} s^k + O(s^{m+1}) = \sum_{k=0}^{m} \frac{a_k}{k!} s^k + \alpha s^{m+1} + O(s^{m+2}).$$

For $\beta \in \mathbb{C}$ we have, using the Taylor expansion of the exponential series, that

$$\exp(\beta s^{m+1}) = 1 + \beta s^{m+1} + O(s^{2m+2}). \tag{1}$$

Hence,

$$\exp\left(\sum_{k=0}^{m} b_k s^k + \beta s^{m+1}\right) = \exp\left(\sum_{k=0}^{m} b_k s^k\right) \exp(\beta s^{m+1})$$

For the first factor on the right hand side we use the induction hypothesis and for the second factor we use (1). We obtain

$$\begin{aligned}
&\exp\left(\sum_{k=0}^{m} b_k s^k + \beta s^{m+1}\right) \\
&\quad = \left(1 + \beta s^{m+1} + O\big(s^{2m+2}\big)\right)\left(\sum_{k=0}^{m} \frac{a_k}{k!} s^k + \alpha s^{m+1} + O\big(s^{m+2}\big)\right) \\
&\quad = \sum_{k=0}^{m} \frac{a_k}{k!} s^k + (\beta a_0 + \alpha) s^{m+1} + O\big(s^{m+2}\big). \qquad (2)
\end{aligned}$$

Let $b_{m+1} = \beta$ be the solution of the equation

$$\beta a_0 + \alpha = \frac{a_{m+1}}{(m+1)!},$$

which exists since $a_0 \neq 0$. Substituting this in (2) we obtain the desired conclusion.

b) We will prove the slightly stronger statement, namely, that the set of such τ's is unbounded in $\mathbb{R}$. Pick r sufficiently small such that

$$D = \{s \in \mathbb{C} : |s - \sigma| \leq r|\} \subsetneq \{s \in \mathbb{C} : \frac{1}{2} < \text{Re}(s) < 1\}.$$

For instance, one can choose any r in the interval $\big(0, \min(\sigma - \frac{1}{2}, 1 - \sigma)/2\big)$. By hypothesis, g does not vanish on D and g is holomorphic on D and continuous on its boundary. Hence $g \in \mathcal{H}(D)$. By Voronin's Universality Theorem (Theorem 3.2.3 in the notes) we get that there exists $\tau_k \in \mathbb{R}$ such that

$$\sup_{s \in D} |\zeta(s + i\tau_k) - g(s)| < \frac{\varepsilon r^k}{k!}.$$

Note that in the above equation, on the left hand side, we take the supremum of a continuous function over a compact domain. Hence the supremum is achieved. We indeed proved that

$$\max_{|s| \leq r} |\zeta(s + \sigma + i\tau_k) - g(s)| < \frac{\varepsilon r^k}{k!}.$$

Pick $0 \leq \ell \leq n - 1$ such that $\frac{\varepsilon r^\ell}{\ell!}$ is minimal. Choosing $\tau = \tau_\ell$ then gives the existence of $\tau \in \mathbb{R}$ with the desired property.

Let $S_1 \subset \mathbb{R}$ be the set of such τ's. So far we have proven that $S_1 \neq \emptyset$. Suppose S_1 is bounded. Pick $\tilde{\tau}_1 \in S_1$. Replacing ε with $\frac{\varepsilon}{2}$ we get that the analogous set S_2 is also nonempty. Moreover $S_2 \subset S_1$. Pick $\tilde{\tau}_2 \in S_2$. Continue inductively. We obtain a sequence $\tilde{\tau}_k \in S_k \subset S_1$ such that

$$\max_{|s| \leq r} |\zeta(s + \sigma + i\tilde{\tau}_k) - g(s)| < \frac{\varepsilon r^i}{2^{k-1} i!} \tag{3}$$

for all $0 \leq i \leq n - 1$. Then (3) implies that, for all s with $|s| \leq r$, we have

$$|\zeta(s + \sigma + i\tilde{\tau}_k) - g(s)| < \frac{\varepsilon r^i}{2^{k-1} i!} \tag{4}$$

for all $0 \leq i \leq n - 1$ and all $k \geq 1$. Since $\tilde{\tau}_k \in S_1$ and S_1 is bounded, we get that there exists a subsequence $\tilde{\tau}_{i_k}$ which converges to some limit $T \in \mathbb{R}$. Using this in (4) and taking $i_k \to \infty$, we immediately obtain that $\zeta(s + \sigma + iT) = g(s)$ for all s with $|s| \leq r$. But then, by identity theorem, $\zeta(s + \sigma + iT)$ and $g(s)$ would coincide everywhere. We have thus obtained a contradiction and therefore S_1 is unbounded.

c) We again prove the slightly stronger statement, that the set of such τ's is unbounded in $\mathbb{R}$. We have, by Cauchy's formula, that for a holomorphic function f

$$f^{(k)}(0) = \frac{k!}{2\pi i} \oint_{|s|=r} \frac{f(s)}{s^{k+1}} ds.$$

Siehe nächstes Blatt!

We use part b). Pick $M > 0$ arbitrarily large. There exists τ with $|\tau| > M$ such that

$$\max_{|s|\le r} |\zeta(s+\sigma+i\tau) - g(s)| < \frac{\varepsilon r^k}{k!}$$

for all $0 \le k \le n-1$. We prove that τ satisfies the desired condition. Choose $f(s) = \zeta(s+\sigma+i\tau) - g(s)$. We get that

$$\begin{aligned}
|\zeta^{(k)}(\sigma+it) - g^{(k)}(0)| &= \left| \frac{k!}{2\pi i} \oint_{|s|=r} \frac{\zeta(s+\sigma+i\tau) - g(s)}{s^{k+1}} ds \right| \\
&\le \frac{k!}{2\pi} \oint_{|s|=r} \left| \frac{\zeta(s+\sigma+i\tau) - g(s)}{s^{k+1}} \right| ds \\
&= \frac{k!}{2\pi r^{k+1}} \oint_{|s|=r} |\zeta(s+\sigma+i\tau) - g(s)| ds \\
&\le \frac{k!}{2\pi r^{k+1}} \oint_{|s|=r} \frac{\varepsilon r^k}{k!} ds \\
&= \frac{\varepsilon}{2\pi r} \oint_{|s|=r} 1 ds = \varepsilon
\end{aligned}$$

for all $0 \le k \le n-1$. Since $|\tau| > M$ and M was picked arbitrarily large, the conclusion follows.

d) We will prove the stronger statement that given $\varepsilon > 0$ and a point $a \in \mathbb{C}^n$, the set $\{t \in \mathbb{R} : |\varphi_\sigma(t) - A| < \varepsilon\}$ is unbounded in $\mathbb{R}$. Pick a point $a = (a_0, \dots, a_{n-1}) \in \mathbb{C}^n$ and $\varepsilon > 0$. Also, pick $M > 0$ arbitrarily large. We want to show that there exists $t \in \mathbb{R}$ with $|t| > M$ such that $|\varphi_\sigma(t) - a| < \varepsilon$. Use part a) on the sequence $(a_0, \dots, a_{n-1})$ and define

$$g(s) = \exp\left(\sum_{k=0}^{n-1} b_k s^k \right)$$

with $g^{(k)}(0) = a_k$ for $0 \le k \le n-1$. From part c) applied to g, we get that there exists $\tau \in \mathbb{R}$ with $|\tau| > M$such that $|\zeta^{(k)}(\sigma+i\tau) - a_k| < \frac{\varepsilon}{\sqrt{n}}$. But then we immediately get $|\varphi_\sigma(\tau) - a| < \varepsilon$. Since $|\tau| > M$ and since M was chosen arbitrarily large, we are done.

2. We proceed in two steps. In the first step, we show that if F is a continuous function such that

$$F(\zeta(s), \zeta^{(1)}(s), \dots, \zeta^{(n-1)}(s)) = 0$$

identically for $s \in \mathbb{C}$, then F is the trivial map. Assume by contradiction that there exists some $a \in \mathbb{C}^n$ for which $F(a) \ne 0$. Since F is continuous, it follows that there exists a neighbourhood U of a and $\varepsilon > 0$ such that

$$|F(z)| > \varepsilon$$

Bitte wenden!

for all $z \in U$. Choose some σ with $\frac{1}{2} < \sigma < 1$. The previous exercises ensures that there exists $\tau \in \mathbb{R}$ such that $\varphi_\sigma(\tau) \in U$. But then $|F(\varphi_\sigma(\tau))| = 0$ which contradicts $|F(\varphi_\sigma(\tau))| > \varepsilon$. Hence F is trivial.

For the second step, we may assume without loss of generality that F_0 is not the trivial map. As before, there exists an open ball U and $\varepsilon > 0$ such that $|F_0(z)| > \varepsilon$ for all $z \in U$. Denote by M the maximum of all indices m for which

$$\sup_{z \in U} |F_m(z)| \neq 0$$

From before we have that such M exists and $M \geq 0$. If $M = 0$, then we proceed as in the first step. If $M > 0$, we choose a subset $V \subseteq U$ such that

$$\inf_{z \in V} |F_M(z)| > \varepsilon.$$

The existence of such V is ensured by the continuity of F_M. From the proof of the previous exercise, there exists a sequence t_j with $|t_j| \to \infty$ such that $\varphi_\sigma(t_j) \in V$. Set

$$R = \max_{0 \leq i \leq M} \sup_{z \in V} |F_i(z)|.$$

But then by the triangle inequality we have that

$$\begin{aligned} \left| \sum_{k=0}^{N} (\sigma + it_j)^k F_k(\varphi_\sigma(t_j)) \right| &= \left| \sum_{k=0}^{M} (\sigma + it_j)^k F_k(\varphi_\sigma(t_j)) \right| \\ &\geq |\sigma + it_j|^M |F_M(\varphi_\sigma(t_j))| - \left| \sum_{k=0}^{M-1} (\sigma + it_j)^k F_k(\varphi_\sigma(t_j)) \right| \\ &\geq |\sigma + it_j|^M |F_M(\varphi_\sigma(t_j))| - \sum_{k=0}^{M-1} |\sigma + it_j|^k |F_k(\varphi_\sigma(t_j))| \\ &> \varepsilon |\sigma + it_j|^M - R \sum_{k=0}^{M-1} |\sigma + it_j|^k \qquad (5) \end{aligned}$$

Because $|t_j| \to \infty$ we immediately get that $|\sigma + it_j| \to \infty$. But then we have that

$$\left| \sum_{k=0}^{N} (\sigma + it_j)^k F_k(\varphi_\sigma(t_j)) \right| \to \infty$$

since the last term in (5) is a polynomial in $|\sigma + it_j|$ with leading coefficient strictly greater than 0. This proves the statement.