ANALYSIS III FOR CIVIL ENGINEERS-LECTURE 1

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Zusammenfassung. First definitions are given and the classification of 2nd order linear PDEs into Parabolic, Elliptic, Hyperbolic is explained.

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1. First definitions and examples

1.1. **What is a partial differential equation?** Let us first fix notation. Let Ω be an open subset of $\mathbb{R}^n$. For a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, all the following notations are commonly used in order to denote the partial derivative with respect to x_i:

$$\frac{\partial f}{\partial x_i}, \frac{\partial}{\partial x_i} f, \partial_{x_i} f, f_{x_i}$$

We will mainly use the last notation, f_{x_i}, but it is important to know them all in order to be able to read other texts. Similarly,

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{ttt} = \frac{\partial^3 f}{\partial t^3}.$$

A *Partial Differential Equation*[1] or in short PDE, is an equation involving partial derivatives of some unknown function (which is called *the dependent variable*). In contrast to Ordinary Differential Equations[2] (ODE for short) where the unknown function is a function of one variable, in a PDE the unknown function depends on *several* variables. A *solution* for a PDE is a function satisfying the relations inscribed by the equation.

Before giving more definitions let us get the ball rolling with some examples.

1.2. **The simplest example.** Arguably, the simplest ODE is $f' = 0$. Its solutions are $f = C$ where C is an *arbitrary* constant. Similarly, the simplest PDE one can think of is

$$\frac{\partial u(x, y)}{\partial x} = 0.$$

This just means that u is independent of x so its solutions are *arbitrary functions* of y. That is $u(x, y) = f(y)$ for some arbitrary function f depending only on y.

[1] Partielle Differentialgleichung, Abkürzung PDG oder PDGL

[2] Gewöhnliche Differentialgleichung Abkürzung GDGL

Remark 1.1. We can already learn two important things from this simple example. The first is that arbitrary functions play the same role that arbitrary constants play in the study of ODE's. The second is that typically, without posing more conditions on the solution, the space of the solution for a given PDE is *huge*.

1.3. **Playing with the one-dimensional Heat Equation.** The following PDE is called the one-dimensional Heat equation:

$$u_t = u_{xx}. \tag{1.1}$$

We will explain in detail later why it is named like that. For now let's just say that t stands for time, and x is a spatial[3] coordinate. The zero function $u(x,t) \equiv 0$ is evidently a solution for (1.1) and the simplest non-zero solution that comes to mind is $u(x,t) = \frac{1}{2}x^2 + t$. Indeed, $u_{xx} = 1$ and $u_t = 1$. Can you find more polynomial solutions?

Exercise 1.2. A polynomial satisfying Equation (1.1) is called heat-polynomial or caloric polynomial. Can you find other such polynomials?

Exercise 1.3. Find more solutions. You may want to try solutions of the form e^{ax+bt} for $a, b \in \mathbb{C}$.

<u>Solution:</u> We did this in class. Substituting $u(x,t) = e^{ax+bt}$ in Equation (1.1) we get

$$u_t = be^{ax+bt} = a^2 e^{ax+bt} = u_{xx}$$

so $u(x,t) = e^{ax+bt}$ is a solution of Equation (1.1) if and only if $b = a^2$. In other words, any function of the form

$$u(x,t) = e^{ax+a^2 t}, \quad a \in \mathbb{C}$$

if a solution for the PDE (1.1).

Remarks 1.4.
(1) The bold attempt to use functions of the form $u(x,t) = e^{ax+bt}$ and hope for the best is usually called *Ansatz* (also in non German-speaking countries). This approach is very common in this field of research as after arriving to a possible solution one may check if it solves the problem or not.
(2) Sometime the Ansatz will be just a simplifying assumption. For example, as we will see, a common simplifying assumption is that the solution has separated variables, that is, $u(x,t)$ is of the form

$$u(x,t) = X(x)T(t)$$

[3]meaning related to space.

where $X(x)$ and $T(t)$ are functions of one variable. The solutions we found above are such solutions:

$$u(x,t) = e^{ax+a^2t} = e^{a^2t}e^{ax}.$$

Just in order to impress upon the reader how much mathematics one PDE can generate, one can take a look at this book [Can84] whose topic is equation (1.1). We will use equation (1.1) in order to introduce some important mathematical tools.

2. General Classification of PDEs

[4]In this course we will mainly learn methods to study the following class of PDEs:

2nd order linear equations with constant coefficients in small number of variables

Thus we begin by defining the (colored) notions appearing in this title:

2.1. **The order of a PDE.** The order of the highest derivative appearing in the equation is called the *order* of the PDE. For example, the Beam Equation

$$u_{tt} = -\alpha u_{xxxx} \tag{2.1}$$

has order four, while Equation (1.1), the 3-dimensional Wave Equation

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right), \tag{2.2}$$

and equation

$$u_{xy} = 0$$

all have order 2. Do not get confused with degree of polynomials! The following equation

$$u_x^4 + u_y u^{18} = 0$$

has order 1.

2.2. **The number of variables.** This is the numbers of independent variables of the unknown function u. For example equation (1.1) has 2 variables, equation (2.2) has 4 variables and equation (2.1) has 2 variables. The only confusion that may arise is that as the above examples show the n-dimensional Heat/Wave equation has in fact $n+1$ variables as n stands only of the spatial (i.e. space-related) variables and there is another time variable t.

[4]In this section we loosely follow [Far93, Ch.1] and [KKN11, §12.1].

2.3. **Linear PDEs.** One should think on PDE as equation relating the dependent variable u and its partial derivative. A PDE is called *linear* if the dependent variable u and its partial derivatives appear in a linear manner. This means that u and its partial derivatives are not multiplied, squared, taken square root of, etc. For understanding this notion, one can also think about u and its partial derivatives as variables. Then a linear PDE is a linear equation in these variables.

Remark 2.1. As the examples below show, the 'scalars' or 'coefficients' can be functions (and not just constants) of the independent variables.

These examples are from [Far93, Ch.1]

(1) $u_{tt} = e^{-t}u_{xx} + \sin t$ is linear. Note that the coefficients multiplying u_{xx} and the constant function 1 are e^{-t} and $\sin t$ respectively.
(2) $uu_{xx} + u_t = 0$ is **not** linear as u and u_{xx} are multiplied.
(3) $u_{xx} + yu_{yy} = 0$ is linear. The scalars multiplying u_{xx} and u_{yy} are 1 and y respectively.
(4) $xu_x + yu_y + u^2$ is **not** linear as u is multiplied with itself.

We are now in position to further classify the most important class of PDEs for our course.

3. Further classification of 2nd order linear equations

The class of 2nd order linear PDEs in two variables is the class of equations of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{3.1}$$

where A, B, C, D, E, F, G are functions in x and y. If all A, B, C, D, E, F, G are constants (respectively not all are constants) then equation (3.1) is said to have *constant coefficients* (respectively *non-constant coefficients*).

[5] Equation (3.1) is called

$$\begin{array}{rcl} \text{homogeneous} & \text{if} & G = 0 \\ \text{non-homogeneous} & \text{if} & G \neq 0 \end{array}$$

The importance of this notion lies in:

Proposition 3.1 (The superposition principle- first form)**.** *If u_1 and u_2 are solutions for a homogeneous linear equation Φ and α_1, α_2 are arbitrary constants, then*

$$\alpha_1 u_1 + \alpha_2 u_2$$

is also a solution Φ. In other words, the space of solutions of a linear homogeneous equation is a vector space.

[5]The following definition and proposition were not discussed in class yet-we will talk about it next week

Here are some examples to practice these notions: the examples above

$$u_{tt} = e^{-t}u_{xx} + \sin t$$

and

$$u_{xx} + yu_{yy} = 0$$

are in the class of 2nd order linear equations. Both have non-constant coefficients. The first is non-homogeneous (because of $\sin t$) and the second is homogeneous. Equation (1.1) is homogeneous with constant coefficients.

3.1. **Parabolic, Elliptic and Hyperbolic equations.** Equation of the form (3.1) is said to be

$$\begin{array}{ccc} \text{Parabolic} & \text{if} & B^2 - 4AC = 0 \\ \text{Hyperbolic} & \text{if} & B^2 - 4AC > 0 \\ \text{Elliptic} & \text{if} & B^2 - 4AC < 0 \end{array}$$

The term Parabolic/Hyperbolic/Elliptic is also called the *type* of the equation.

Remarks 3.2.
(1) Note that the type of Equation (3.1) does not depend on the coefficients D, E, F, G.
(2) If A, B, C are non-constants, i.e., they are functions of x and y, they may have different type for different values of x and y (see an example below).
(3) We will see and explain later that the type of the equation stays invariant under coordinate change.
(4) Why is this notion important? It turns out that the type of the equation tells us quite a lot about it. For example, equation of the same type usually model similar phenomenas, their solutions share similar properties, one can apply similar methods in order to solve them, etc.
(5) <u>Technical remark</u>- Some books and teachers use $\left(\frac{B}{2}\right)^2 - AC$ or $AC - \left(\frac{B}{2}\right)^2$ in order to determine the type. Even worse, some assume that the coefficient of u_{xy} in Equation (3.1) is $2B$ and evaluate $AC - B^2$ for determining the type. So be alert to this issue when you are using different notes and books.

3.2. **Actual 'recipe' for finding the type of a given equation.** This explanation will be reviewied during your first exercise class. We will exemplify the steps below using the equation $u_{ss} = u_{pp}$ for $u = u(s, p)$.

(1) rearrange the equation by moving all the terms (or at least the terms with the second order partial derivatives) to one side (it does not matter which one). (e.g. change $u_{ss} = u_{pp}$ into $u_{ss} - u_{pp} = 0$ or $u_{pp} - u_{ss} = 0$.)
(2) Find out the coefficient of the *mixed* partial derivative. This is B. Note that the names of your variable can vary but we assume that there are

only two variables so there is a unique mixed partial derivative of order 2. In our example $B = 0$ as the coefficient of u_{sp} is 0.

(3) Choose one of the variables to be the first and the other to be second, so you can see who is A and who is C. It doesn't matter which variable you choose as the first or second.

3.3. **Three main examples.**
These examples are the usual representatives of each class of equations. We will study these equations and their variants in details during our course.
The 1-dimentional Heat Equation $u_t = \alpha u_{xx}, \alpha > 0$.
Here $B = 0$ as there is no mixed term and $A = \alpha, C = 0$ (or if we reorder the variables, $A = 0, C = \alpha$). In any case $B^2 - 4AC = 0$ so it is a *parabolic* equation.
The 1-dimensional wave equation $u_{tt} = c^2 u_{xx}$.
We rewrite $u_{tt} - c^2 u_{xx} = 0$ so $B = 0$ and $A = 1, C = -c^2$. Therefore $B^2 - 4AC = 4c^2 > 0$ so it is a *hyperbolic* equation.
The 2-dimensional Laplace equation $u_{xx} + u_{yy} = 0$.
Here $B = 0$ and $A = 1, C = 1$. Therefore $B^2 - 4AC = -4 < 0$ so it is a *elliptic* equation.

3.4. **Further examples.**
The 'disguised' wave equation $u_{\xi\eta} = 0$. We mention, without proof for now, that this 1-dimensional wave equation $u_{tt} = c^2 u_{xx}$ under the change of coordinates $\xi = x + ct, \eta = x - ct$. By the remark above, this means that both equations have the same type, so this equation is Hyperbolic. Checking this directly, we see that $B = 1, A = C = 0$ so $B^2 - 4AC = 1 > 0$ so it is indeed Hyperbolic.
The Euler-Tricomi Equation

$$yu_{xx} + u_{yy} = 0. \tag{3.2}$$

Here $A = y, B = 0, C = 1$ so $B^2 - 4AC = -4y$. Therefore, Equation (3.2) is

$$\begin{array}{ccc} \text{Parabolic} & \text{if} & y = 0 \\ \text{Hyperbolic} & \text{if} & y < 0 \\ \text{Elliptic} & \text{if} & y > 0 \end{array}$$

3.5. **Solution to the Clicker questions.** These were taken from the 2013 exam of D–MATL, D–MAVT in analysis III.

Exercise 3.3. Determine the type of

$$u_{xx} + 2u_{xy} + u_{yy} + 3u_x + xu = 0 \tag{3.3}$$

$$u_{xx} + 2u_{xy} + 2u_{yy} + u_y = 0 \tag{3.4}$$

$$u_{xx} + 8u_{xy} + 2u_{yy} + e^x u = 0 \tag{3.5}$$

Solution 3.4. Here there is no need to rearrange the equation. For Equation (3.3), $A = 1, B = 2, C = 1$, $B^2 - 4AC = 0$ so it is parabolic. For Equation (3.4), $A = 1, B = 2, C = 2$, $B^2 - 4AC = -4$ so it is elliptic. For Equation (3.5), $A = 1, B = 8, C = 2$, $B^2 - 4AC = 56$ so it is hyperbolic.

Literatur

[Can84] J. R. Cannon, *The one-dimensional heat equation*, Cambridge University Press, 1984. ↑3

[Far93] S. J Farlow, *Partial differential equations for scientists and engineers*, Courier Corporation, 1993. ↑3, 4

[KKN11] E. Kreyszig, H. Kreyszig, and E. J. Norminton, *Advanced engineering mathematics*, Tenth, Wiley, Hoboken, N.J., 2011. ↑3

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