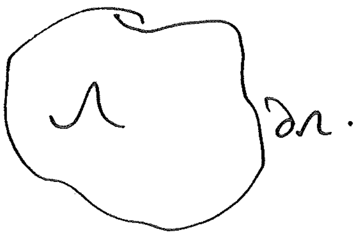
6. The poisson Equation:

Let $\Omega \subseteq \mathbb{R}^n$ $(n=1,2,3)$ be a domain (open set) with boundary $\partial\Omega$ (see figure 1)



Then the unknown u in any PDE ~~(scalar or vector~~ is a (scalar or vector) function $u : \Omega \mapsto \mathbb{R}^m$. We denote its derivatives as:

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$$

$D^2 u$ is the second derivative .... and

$D^k u$ denotes the k-th derivative.

then the most general form of a PDE (partial differential equation) is given by

$$F(x, u, \nabla u, D^2 u, \dots, D^k u) = 0 \quad \text{— (6.1)}$$

for some nonlinear function F.

The task of solving a PDE reduces to finding a function u, given a complicated "nonlinear" relationship. between a combination of its derivatives.

A large number of PDEs appear in models in physics and engineering. We will study some of the most important PDEs that arise in engineering and design efficient methods to solve them.

6.1.A: Derivation of the Poisson's equation:

One of the most important PDEs is given by the Poisson's equation,

$$-\Delta u = f \quad \text{— (6.2)}$$

Here, f is a source (given) function and $u: \Omega \mapsto \mathbb{R}$ with

$$\Delta = \sum_{i=1}^{n} u_{x_i x_i} \quad \text{is the Laplace operator,}$$

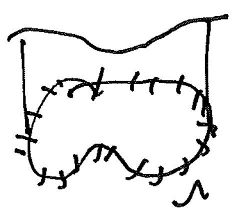
i.e., the trace of $D^2 u$.

We will provide the following derivation of the Poisson's equation (6.2).

6.1.2: A variational principle.

In many situations in physics, the model boils down to choosing one configuration of a system among many possible configurations. The sought for configuration is usually a minimizer (maximizer) for some variational problem as in the following example.

Consider an elastic body defined on domain Ω. For instance a membrane clamped at the boundary (see fig 2)



— membrane

The unknoown is the displacement $u = u(x), x \in \Omega$.

Clamping at the bamdary implies the "boundary condition"

$$u\Big|_{\partial\Omega} \equiv 0 \quad - \quad (6.3)$$

The total energy (elastic energy) can be modeled by.

$$J(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} uf\, dx \quad - \quad (6.4)$$

~~kneti~~

with some load $f(x)$.

The sought for configuration is a minimizer of the energy J of (6.4) (J is also termed the Dirichlet energy)

As in calculus, we can calculate a minimizer, by. calculating the so-called Euler_lagrange equations,

$$J'(u) = 0 = \lim_{\tau \to 0} \cdot \frac{J(u+\tau v) - J(u)}{\tau}$$

we perform the following (formal) calculation,

$$\frac{J(u+\tau v) - J(u)}{\tau} = \frac{1}{\tau}\left(\frac{1}{2}\int |\nabla(u+\tau v)|^2 dx - \int |\nabla u|^2 dx - \tau \int f v\, dx\right)$$

using: $|\nabla w|^2 = \langle \nabla w, \nabla w\rangle$ ($\langle , \rangle$ - inner product in $\mathbb{R}^n$), We see that

$$\frac{J(u+\epsilon v)-J(u)}{\epsilon} = \frac{1}{\epsilon}\left[\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \epsilon\int \langle \nabla u, \nabla v\rangle\,dx + \frac{\epsilon^2}{2}\int |\nabla v|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \epsilon\int f v\,dx\right]$$

$$= \cancel{\epsilon}\int_\Omega \langle \nabla u, \nabla v\rangle\,dx + \frac{\epsilon}{2}\int |\nabla v|^2\,dx - \int f v\,dx.$$

$$\therefore \lim_{\epsilon\to 0}\frac{J(u+\epsilon v)-J(u)}{\epsilon} = \int_\Omega \langle \nabla u, \nabla v\rangle\,dx$$

Hence,

$$J'(u) = \int_\Omega \langle \nabla u, \nabla v\rangle\,dx - \int_\Omega f v\,dx,$$ for all v such that $v \equiv 0$ on $\partial\Omega$.

$\therefore$, the Euler-Lagrange equations are,

$$\int_\Omega \langle \nabla u, \nabla v\rangle\,dx - \int_\Omega f v\,dx = 0$$

By integrating by parts ~~and~~ (using the Green's formula), the above is,

$$-\int_\Omega v\,\Delta u\,dx - \int_\Omega v f\,dx. + \int_{\partial\Omega} v\frac{\partial u}{\partial\nu}\,ds(x) = 0$$

Here $\frac{\partial u}{\partial \nu} = \nabla u\cdot\nu$ and ν is the outward unit normal to $\partial\Omega$.

Using the fact that any admissible configuration is clamped at the boundary i.e. $v \equiv 0$ on $\partial\Omega$, the above reduces to:

$$\int_{\Omega} (-\Delta u - f)\, v \, dx = 0, \qquad \forall v$$

hence,

$$-\Delta u = f \text{ and}$$
$$u \equiv 0 \text{ on } \partial\Omega, \qquad (6.5)$$

thus deriving the Poisson's equation (6.2).

The Poisson's equation can also be derived ~~from~~ as a steady state of the heat equation (see chapter 8) as well as the potential flow equations of fluid dynamics.

6.2 The Poisson equation in one-space dimension

In one space dimension, we consider the domain $\Omega = (0,1)$, the Poisson equation (6.5) assumes the form,

$$-u''(x) = f(x), \quad \forall x \in (0,1) \qquad (6.6).$$
$$u(0) = u(1) = 0$$

~~We can find an~~ Note that (6.6) is an example of a two-point boundary value problem (BVP) for ODEs.

We can find an explicit formula for the solutions of (6.6).

To this end; we write using fundamental theorem of integral Calculus that;

$$u'(y) = c_2 + \int_0^y u''(z)\, dz \qquad (c_2 \text{ is a constant})$$

$$= c_2 - \int_0^y f(z)\, dz \qquad (\text{from } 6.6).$$

and

$$u(x) = c_1 + \int_0^x u'(y)\, dy$$

$$= c_1 + c_2 x - \int_0^x \int_0^y f(z)\, dz\, dy \qquad (6.7)$$

Let $F(y) = \int_0^y f(z)\, dz$, then $F'(y) = f(y)$.

By integration by parts.

$$\int_0^x F(y)\, dy = \int_0^x y' F(y)\, dy = xF(x) - \int_0^x y f(y)\, dy$$

$$= x \int_0^x f(y)\, dy - \int_0^x y f(y)\, dy$$

$$= \int_0^x (x-y) f(y)\, dy.$$

Hence, (6.7) can be rewritten as,

$$u(x) = c_1 + c_2 x - \int_0^x (x-y) f(y)\, dy.$$

We determine the constants C_1, C_2 from the boundary condition in (6.6),

$$0 = u(0) = C_1 \Rightarrow \quad \text{and,}$$

$$0 = u(1) = C_2 - \int_0^1 (1-y) f(y)\, dy$$

$$\Rightarrow \quad C_2 = \int_0^1 (1-y) f(y)\, dy.$$

Hence,

$$u(x) = \int_0^1 x(1-y) f(y)\, dy - \int_0^x (x-y) f(y)\, dy. \quad \text{— (6.8)}$$

<u>Define</u>

$$G(x,y) = \begin{cases} y(1-x) & 0 \le y \le x \\ x(1-y) & x \le y \le 1 \end{cases} \quad \text{— (6.9)}$$

then we can check that (6.8) is equivalent to:

$$u(x) = \int_0^1 G(x,y) f(y)\, dy. \quad \text{— (6.10)}$$

The function G in (6.9) is termed as a Green's function for the 1-D Poisson and provides an explicit solution formula equation (6.6).

6.2.1: limitations of the Green's function ~~limitation~~ Representation

The explicit formula (6.10) is not very useful as

1. The integral in (6.10) is not possible to evaluate exactly for complicated ~~f~~ source ~~fund~~ (load) functions f.

A numerical quadrature rule needs to be used.

2. A slight perturbation in the form of the Poisson's equation ~~with~~ may lead to us not being able to find a solution formula like (6.10). For instance, in many applications, a modified form of the Poisson's equation,

$$-(a(x)\,u')' + b(x)\,u' + c(x)\,u = f, \quad \forall x \in (0,1)$$

$$u(0) = u(1) = 0 \qquad \text{— (6.11)},$$

with coefficients a, b, c, arises.

It is not possible to find explicit formulas even for simple cases such that $b(x) \equiv 0$ and $c(x) \equiv c$.

3. Green's function representations are not available for the Poisson equation, ~~even~~ even 2- (and 3-) dimensional forms of the Poisson equation, except for very simple domains, such as the ~~square~~ Ball.

6.3: Finite difference methods

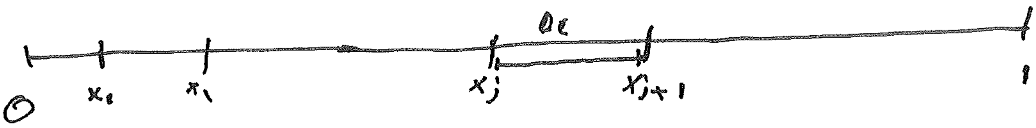
Given the limitations of explicit solution formulas, we use numerical methods. The simplest numerical method to approximate the 1-D Poisson equation (6.6) is a finite difference method.

6.3.1: Discretizing the domain.

Let $\Delta x > 0$, $N+1 = 1/\Delta x$. Then we can discretize the domain $(0,1)$ into $(N+2)$ points i.e.

$x_0 = 0$, $x_{N+1} = 1$ and $x_j = j\,\Delta x$,

See figure ...



6.3.2: Discretizing the Derivatives:

Our aim will be to approximate the function $u(x)$, that solves (6.6), with point values i.e.

$$U_j \approx u(x_j)$$

Similarly

$$f_j \approx f(x_j)$$

Hence, we need to approximate the derivatives that appear in (6.6) with finite differences. We use a simple centered approximation of the second derivative in (6.6) i.e.

$$u''(x_j) \approx \frac{U_{j+1} - 2U_j + U_{j-1}}{\Delta x^2} \qquad (6.12)$$

Ex: Check that for sufficiently smooth functions $u(x)$

$$\left| u''(x_j) - \frac{u(x_j+\Delta x) - 2u(x_j) + u(x_j-\Delta x)}{\Delta x^2} \right| \le C\Delta x^2. \qquad (6.13)$$

6.3.3: The finite difference scheme

A finite difference scheme for approximating (6.6) is given by: $\forall\ j = 2, \dots, N-1$

$$-U_{j-1} + 2U_j - U_{j+1} = \Delta x^2 f_j$$

using the boundary condition, $U_0 = u(0) = 0$, we see that

$$2U_1 - U_2 = \Delta x^2 f_1$$

Similarly using the boundary condition, ie

$$u_{N+1} = u(x_{N+1}) = u(1) = 0,$$

$$-u_{N-1} + 2u_N = 0$$

Writing the vector U as

$$U = [u_1, u_2, ---, u_N]$$

and $F = \Delta x^2 [f_1, f_2, ---, f_N]$

We observe that the above finite difference scheme can be recast as the following matrix equation,

$$AU = F \quad \text{— (6.14)}$$

with the $N \times N$ matrix A defined by

$$A = \begin{bmatrix} 2 & -1 & 0 & \cdots & & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & \cdots & & & & \\ 0 & & -0 & -1 & 2 \; -1 & 0 \\ \cdots & & & & 0 \; -1 & 2 \end{bmatrix}$$

6.3.4 : Solving the Matrix Equation

Note that a numerical scheme for approximating the Poisson's equation (6.6) reduces to solving a matrix equation (6.14). Observe that A is a tridiagonal, diagonally dominant matrix. Hence it is invertible. (6.14) can be solved using methods learnt in numerical linear algebra.

6.3.5: Numerical results:

See accompaning slides.

In fact, using discrete version of the Green's function representation, one can prove the following ~~error~~ Stability estimate,

$$||U^{\Delta x}||_\infty \leq \frac{1}{8} ||f^{\Delta x}||_\infty$$

Here for any given Δx,

$U^{\Delta x} = [u_1, u_2, ---, u_N]$ and

$f^{\Delta x} = [f_1, f_2, ---, f_N]$

with $U^{\Delta x}$ solving the matrix equation 6.14.

Furthermore:

$$||U^{\Delta x}||_\infty = \max_{1 \leq j \leq N} |u_j|$$

$||f^{\Delta x}||_\infty$ is analogously defined.

One can also prove the following error estimate,

let $E_j = u(x_j) - u_j^{\Delta x}$

$E^{\Delta x} = [E_1, ---, E_N]$

then $||E^{\Delta x}||_\infty \leq \frac{\Delta x^2}{96} \max_{0 \leq x \leq 1} |f''(x)|$

The above error estimate justifies the observed second order of convergence.

6.3.6 Finite difference Schemes for the 2-D poisson Equation:

If we consider the 2-dimensional version of the Poisson's equation in the unit square i.e. $\Omega = (0,1)^2$;

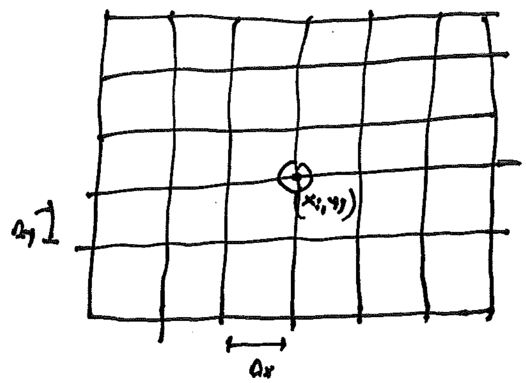
$$-(u_{xx} + u_{yy}) = f \quad \text{on } \Omega = (0,1)^2 \qquad \text{— (6.19)}$$
$$u = 0 \quad \text{on } \partial\Omega$$

then we can formulate a finite difference scheme.

let $\Delta x, \Delta y > 0$, we discretize the square into a set of $(N+1) \times (M+1)$ points. by labeling:

$$x_i = i\Delta x \quad \forall\ 1 \le i \le N. \qquad \Delta x = \frac{1}{N+1}$$
$$x_0 = 0, \quad x_{N+1} = 1$$
$$y_j = j\Delta x \quad \forall\ 1 \le j \le M \qquad \Delta y = \frac{1}{M+1}$$
$$y_0 = 0 \quad y_{M+1} = 1$$

See figure (6.10) for the mesh



Then the aim of a finite difference Scheme is to approximate

$$U_{ij} \approx u(x_i, y_j).$$

Let $\quad f_{ij} = f(x_i, y_j)$

We discretize the Laplacian with a Central difference approximation i.e.,

$$u_{xx}(x_i, y_j) \approx \frac{u(x_i + \Delta x, y_j) - 2u(x_i, y_j) + u(x_i - \Delta x, y_j)}{\Delta x^2}$$

$$\approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2}$$

Similarly $U_{yy}(x_i, y_j) \approx \dfrac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{\Delta y^2}.$

Hence the finite difference approximation ~~is~~ of (6.14) is

$$-\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{\Delta y^2}\right) = f_{ij} \quad \text{--- (6.15)}$$

The boundary conditions are imposed as:

$$U_{0j} = U_{N+1\,j} = 0, \quad \forall\ j = 0, 1, \ldots, M+1$$
$$U_{i0} = U_{iM+1} = 0, \quad \forall\ i = 0, 1, \ldots, N+1$$

For simplicity, let $\Delta x = \Delta y$. Hence, $N = M$.

labeling the vector:

$$U = (U_{11}, U_{21}, \ldots U_{N1}, U_{12}, U_{22}, \ldots, U_{N2}, \ldots\ldots, U_{1N}, U_{2N}, \ldots U_{NN})$$

~~$F = \Delta x^2 (f_{11}, f_{21},$~~

$$F = \Delta x^2 (f_{11}, f_{21}, \ldots f_{N1}, \ldots\ldots\ldots, f_{1N}, f_{2N}, \ldots f_{NN})$$

and $(N^2 \times N^2)$ matrix A as

$$A = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \cdots & \cdots & 0 & -1 & \cdots & 0 \\ \cdots & & & & \vdots & & & & \cdots \\ & & & & \vdots & & & & \\ & & 0 & & -1 & 0 & \cdots & -1\ 4 & -1 \\ & 0 & & & & -1 & \cdots & & -1\ 4 \end{pmatrix}.$$

We can check that (6.15) reduces to the matrix equation.

$$AU = F \quad !!!$$

Finite difference schemes can be defined for other simple two-dimensional domains such as rectangles. However for complex geometries, even for circles, it is not possible to define finite difference schemes. One needs to find an alternative strategy - this is provided by finite element methods, considered in the next chapter.

7. Finite element methods for 1-D Poisson equations

Finite element methods (FEM) are a powerful (and heavily used) alternative to the finite difference methods of the last chapter. We start with a description of these methods in one-space dimension.

Consider the one-dimensional version of the Poisson's equation

$$-u'' = f \quad \text{on} \quad (0,1) \qquad \text{— (7.1)}$$

$$u(0) = u(1) = 0$$

In the last chapter, we derived the Poisson's equation as a the Euler Lagrange equations, corresponding to the solutions of the variational problem;

$$\min_u J(u)$$

with:

$$J(u) = \frac{1}{2}\int_0^1 |u'(x)|^2\,dx - \int_0^1 u(x)\,f(x)\,dx \qquad \text{— (7.2)}$$

we also assumed that $u(0) = u(1) = 0$.

We need to study the above variational problem in some detail.

7.1 Variational principles

The first question concerning (7.2) is: in what set (class) of functions do we seek a minimizer of $J(u)$?

The Boundary conditions impose a restriction on the set of admissible funtions in which a minimizer is sought for. Are there other constraints?

A natural constraint to be restrict u to those functions for which $J(u)$ is well defined. This can be ensured only if

$$\int_0^1 |u'(x)|^2 \, dx < +\infty, \quad — \quad (7.3)$$

and

$$\left| \int_0^1 u(x) f(x) \, dx \right| < +\infty, \quad — \quad (7.4)$$

This ensures that we never subtract infinities.

We will narrow down the set of admissible functions by imposing (7.3) and (7.4).

Define:

~~$H_0^1((0,1)) := \{u : [0,1] \mapsto \mathbb{R} :$~~ $\int$

$$H_0^1((0,1)) := \left\{ u : (0,1) \mapsto \mathbb{R} \text{ such that } u(0) = u(1) = 0 \text{ and } \int_0^1 |u'(x)|^2 \, dx < +\infty \right\}$$

i.e. the class of functions that vanish at the boundary and whose integral of square of derivative is bounded }

also define:

$$\|u\|_{H_0^1(0,1)} := \left(\int_0^1 |u'(x)|^2 \, dx \right)^{1/2}$$

Note that (7.3) automatically holds $\forall \, u \in H_0^1((0,1))$.

$H_0^1((0,1))$ is a prototypical example of a Sobolev space.

Ex 1: $u(x) = x(1-x)$

clearly $u(0) = u(1) = 0$

$$u'(x) = 1-2x$$

$$\int_0^1 |u'(x)|^2\,dx = \int_0^1 |1-2x|^2\,dx \le 2 \quad \text{i.e. } u \in H_0^1((0,1))$$

Ex 2:
$$u(x) = x \text{ if } x \in (0,1/2)$$
$$= \cancel{1/2 - x \text{ if}}$$
$$1-x \text{ if } x \in (1/2,1)$$

Check that $u \in H_0^1((0,1))$

Ex 3:
$$u(x) = x \text{ if } x \in (0,1/2)$$
$$= 0 \text{ if } x \in (1/2,1)$$

As the derivative is not defined at $x = 1/2$, $u \notin H_0^1((0,1))$

~~B~~ By ~~is~~ restricting the set of admissible functions to H_0^1, we automatically satisfy (7.3). What about (7.4), we have the following calculation:

$$\left|\int_0^1 u(x) f(x)\right| dx \le \int_0^1 |u(x)|\,|f(x)|\,dx$$

$$\le \left(\int_0^1 |u(x)|^2 dx\right)^{1/2} \left(\int_0^1 |f(x)|^2 dx\right)^{1/2}$$

(The above follows by the Cauchy-Schwarz inequality.

Define:

$$L^2((0,1)) = \left\{ g : (0,1) \mapsto \mathbb{R} : \int |g(x)|^2 dx < +\infty \right\}$$

and $$\|g\|_{L^2(0,1)} = \left(\int |g(x)|^2 dx \right)^{1/2}.$$

Hence, if we assume that $u, f \in L^2((0,1))$, we see that

$$\left| \int_0^1 u(x) f(x) dx \right| \leq \|u\|_{L^2} \|f\|_{L^2} < +\infty.$$

Thus, (7.2) is well-defined if

$u \in H_0^1((0,1))$ and $f \in L^2((0,1))$.

$u \in L^2((0,1))$

However, there is a further twist to this tale, as can be seen from the following simple calculation;

Let $u(0) = 0$

then by fundamental theorem of calculus,

$$u(x) = \int_0^{*} u'(s)\, ds, \quad \forall\ x \in (0,1)$$

$$\therefore \quad |u(x)| \leq \int_0^x |u'(s)| ds \leq \int_0^1 |u'(s)| ds. \quad \forall\ x \in (0,1).$$

Thus,

$$|u(x)| \leq \int_0^1 1 \cdot |u'(s)| ds \leq \left(\int_0^1 1^2 dx \right)^{1/2} \left(\int_0^1 |u'(s)|^2 ds \right)^{1/2}$$

(Cauchy-Schwartz inequality)

hence

$$|u(x)|^2 \leq \left(\int_0^1 |u'(s)|^2 ds \right) , \quad \forall x \in (0,1).$$

now integrating both sides over $(0,1)$ wrt x, we obtain.

$$\int_0^1 |u(x)|^2 dx \leq \int_0^1 |u'(x)|^2 dx$$

or

$$\|u\|_{L^2((0,1))} \leq \|u\|_{H_0^1((0,1))} \quad \text{—— (7.5)}$$

(7.5) is an example of a Poincarë inequality.

Therefore, ~~Ius~~ just imposing $u \in H_0^1((0,1))$ and $f \in L^2((0,1))$, we can see that the energy functional (7.4) is well defined.

The precise statement of the variational principle is:

~~Find $u \in H$~~

" ~~Given $f \in L^2((0,1))$, find $\bar{u} \in H_0^1((0,1))$, such that:~~
~~$\bar{u}$ minimizes $J(u)$ in (7.4),~~ $\forall u \in H_0^1((0,1))$

" Given $f \in L^2((0,1))$, find $u \in H_0^1((0,1))$, such that u minimizes $J(v)$ in (7.4), $\forall v \in H_0^1((0,1))$ "

i.e. find $u \in H_0^1((0,1))$ such that

$$J(u) = \min_{v \in H_0^1((0,1))} J(v) \quad \text{—— (7.6)}$$

7.3: A Variational formulation.

As is standard in Calculus of Variations, we will compute the minimizer u of (7.6) by the Euler-Lagrange equations i.e.

Find $u \in H_0^1((0,1))$ such that

$$J'(u) = 0 \quad \text{with}$$

$$J'(u) = \lim_{\epsilon \to 0} \frac{J(u+\epsilon v) - J(u)}{\epsilon}, \quad \forall\, v \in H_0^1((0,1))$$

By repeating the formal calculation of chapter 6, we see that u has to satisfy:

$$\int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 v(x)\, f(x)\, dx. \qquad (7.7)$$

$$\forall\, v \in H_0^1((0,1))$$

Several remarks are in order here,

Rem 1: (7.7) is termed as the variational formulation of the 1-D poisson equation. It is also known as the principle of virtual work in structural mechanics.

Rem2: Note that (7.7) is well defined as

$u, v \in H_0^1$. therefore.

$$\int_0^1 u'(x)\, v'(x)\, dx \le \|u\|_{H_0^1} \|v\|_{H_0^1} < +\infty$$

(Cauchy-Schwartz inequality)

$$\int_0^1 v(x)\, f(x)\, dx \le \|v\|_{L^2} \|f\|_{L^2} \quad \text{(Cauchy-Schwartz)}$$

$$\le \|v\|_{H_0^1} \|f\|_{L^2} \quad \text{(Poincare inequality)}$$

$$< +\infty$$

Rem 3: As $u \in H_0^1((0,1))$, we can set $v = u$ in (7.7), then we obtain:

$$\|u\|_{H_0^1}^2 = \int_0^1 |u'(x)|^2\, dx = \int_0^1 u(x) f(x)\, dx$$

$$\leq \|u\|_{L^2} \|f\|_{L^2} \quad \text{(Cauchy-Schwartz)}$$

$$\leq \|u\|_{H_0^1} \|f\|_{L^2} \quad \text{(Poincare)}$$

$$\Rightarrow \quad \|u\|_{H_0^1} \leq \|f\|_{L^2}. \quad \text{— (7.8)}$$

Note that (7.8) is provides a stability estimate on solution of the variational formulation in terms of the data f. !!!

Rem 4: if u solves (7.7) and it is more regular i.e. $u''(x)$ exists and is bounded, then we can integrate by parts in (7.7) to obtain -

$$-\int_0^1 u''(x)\, v(x)\, dx = \int_0^1 v(x) f(x)\, dx$$

$$\Rightarrow \int_0^1 (-u''(x) - f(x))\, v(x)\, dx = 0, \quad \forall v \in H_0^1(\Omega)$$

$$\Rightarrow \quad -u''(x) = f(x)$$

and we recover the pointwise form of the 1-D Poisson equation (7.1). (7.1) is also termed the strong form.

However, the variational form (7.7) is more fundamental as it is directly derived from the variational principle (7.6).

FEM amounts to a discretization of this variational formulation.

7.4: The Finite Element formulation:

Let $V = H_0^1((0,1))$, then the ~~finite element~~ Variational formulation of the one-dimensional Poisson's equation is:

Find $u \in V$; such that

$$\int_0^1 u'(x)\, v'(x)\, dx = \int_0^1 v(x)\, f(x)\, dx \qquad — (7.9)$$

holds $\forall\ v \in V$.

Denote: $$(g,h) = \int_0^1 g(x)\, h(x)\, dx$$

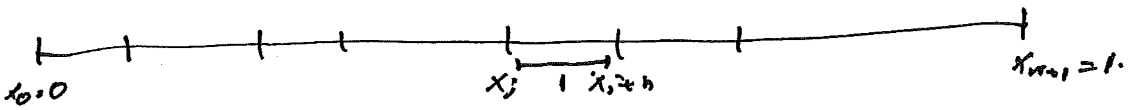
then (7.9) is written concisely as

$$(u', v') = (f, v), \quad \forall\ v \in V. \qquad — (7.10)$$

The finite element method (FEM) replaces the infinite dimensional (function) space V with a suitable finite dimensional subspace V_h and that (7.10) ~~is impos~~ holds on this $V_h \subset V$.

To be more specific, discretize $(0,1)$ exactly as in the finite difference method i.e. given $h > 0$, let N be such that

$$h = \frac{1}{N+1}, \quad \text{let } x_0 = 0 \quad x_{N+1} = 1$$
$$x_j = jh. \qquad \text{See figure (7.1)}$$



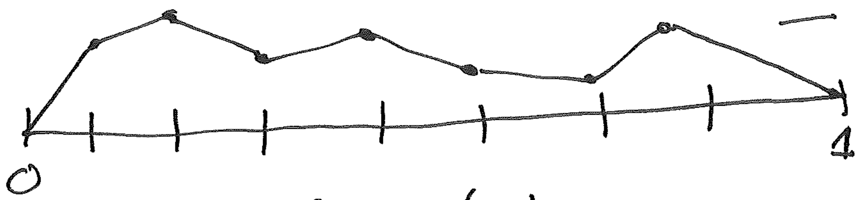
and let

$$V_h = \{ w \text{ is continuous on } (0,1),\ w(0) = w(1) = 0 \text{ and } w\big|_{[(j-1)h,\, jh]} \text{ is linear},\ \forall\ j \in (1, \dots, N+1) \}$$

i.e. V_h consists of all ~~picciew~~ continuous piecewise linear functions on $(0,1)$, with respect to the partition

$$[0,1] = \bigcup_{j=1}^{N+1} [(j-1)h, jh]$$

A typical example of a function w in V_h is shown in figure 7.2



— typical $w \in V_h$:

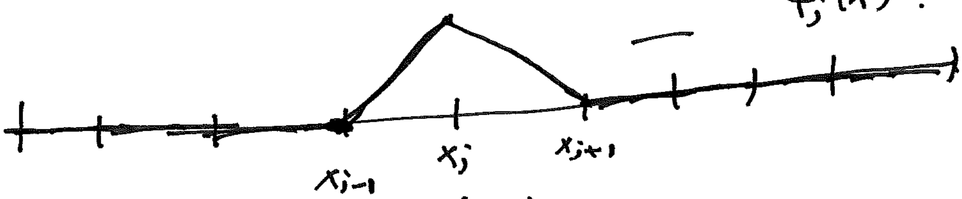
figure (7.2)

Ex 1: Check that $V_h \subset V = H_0^1((0,1))$

Rem: Consider the functions: $\varphi_j(x)$ is piecewise linear, continuous with

$$\varphi_j(x_i) = 1 \quad \text{if} \quad i = j$$
$$= 0, \quad \text{otherwise}.$$

(see figure 7.3 for an example)



— $\varphi_j(x)$.

figure (7.3);

It is easily checked that $\{\varphi_j\}$ form a basis of V_h as any function $w \in V_h$ can be written as:

$$W(x) = \sum_{j=1}^{N} w_j \varphi_j(x) \quad — \quad (7.11)$$

with
$$w_j = w(x_j).$$

Hence, V_h is finite dimensional.

The functions φ_j are termed as "hat" or "tent" functions.

The FEM for the 1-D Poisson equation ~~are~~ consists of the following:

"Find $u_n \in V_n$ such that.

$$(u_n', v') = (f, v) \quad \text{— (7.12)}$$

holds for all $v \in V_n$"

Thus, the finite element approximation u_n is a piecewise linear continuous function, approximating the solution of the Poisson's equation. This should be contrasted to the finite difference method where point values of u are approximated.

7.4.1: Concrete Realization of FEM.

As $v \in V_n$, we know from (7.11) that

$$v = \sum_{j=1}^{N} v_j \, \varphi_j(x)$$

$\therefore$ (7.12) $\Longleftrightarrow$:

$$(u_n', v') = (f, v)$$

$$= \left(u_n', \left(\sum v_j \varphi_j(x)\right)'\right) = \left(f, \sum v_j \varphi_j\right)$$

using linearity of $(\cdot, \cdot)$, we have that

$$\sum_{j=1}^{N} v_j \, (u_n', \varphi_j') = \sum_{j=1}^{N} (f, \varphi_j) \, v_j$$

which holds iff:

$$(u_n', \varphi_j') = (f, \varphi_j), \quad \forall \, 1 \leq j \leq N.$$

— (7.13)

Furthermore as $u_n \in V_n$

$$u_n = \sum_{i=1}^{N} u_i \varphi_i(x)$$

$\therefore$ from (7.13), we have that,

$$\sum_{i=1}^{N} u_i (\varphi_i', \varphi_j') = \cancel{\sum_{j=1}^{N}} (f, \varphi_j).$$

Define the following $N \times N$ matrix

$$A = \{A_{ij}\}_{i,j=1,\dots,N} \quad \text{with}$$

$$A_{ij} = (\varphi_i', \varphi_j') \quad \text{— (7.14).}$$

and the following vectors, $F = \{f_j\}_{j=1}^{N}$

$$f_j = (f, \varphi_j).$$

$$U = \{u_j\}_{j=1}^{N}$$

we see that (7.13) reduces to the matrix equation,

$$AU = F \quad \text{— (7.15).}$$

The matrix A is termed the stiffness matrix

the vector F is termed as the load vector and

and vector U is called the solution vector.

Thus, FEM reduces to a matrix equation. !!!

Clearly, A is a symmetric matrix as:

$$A_{ij} = (\varphi_i', \varphi_j') = \int_0^1 \varphi_i'(x)\, \varphi_j'(x)\, dx$$

$$= \int_0^1 \varphi_j'(x)\, \varphi_i'(x)\, dx = (\varphi_j', \varphi_i') = A_{ji}.$$

Furthermore, let $\omega \in \mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ is the $\mathbb{R}^N$ inner product.

$$\langle A\omega, \omega \rangle = \sum_{i,j=1}^{N} \omega_i A_{ij} \omega_j = \sum_{i,j=1}^{N} \omega_i (\varphi_i', \varphi_j') \omega_j$$

$$= \left(\sum_{i=1}^{N} \omega_i \varphi_i', \sum_{j=1}^{N} \omega_j \varphi_j' \right)$$

Let $\bar{\omega} = \sum_{i=1}^{N} \omega_i \varphi_i$ $\quad = (\bar{\omega}', \bar{\omega}') > 0$ (if $\bar{\omega} \neq 0$)

Thus A is positive-definite:

Given that A is a symmetric, positive definite matrix, we know that A is invertible and linear system (7.15) is solvable.

Thus, the FEM (7.12) is well-defined.

7.4.2 Computing the Stiffness matrix and the Load vector.

We can compute the stiffness matrix explicitly as:

$$\varphi_j'(x) = \frac{1}{h} \text{ if } x \in [(j-1)h, jh]$$

$$= -\frac{1}{h} \text{ if } x \in [jh, (j+1)h].$$

$$= 0, \text{ otherwise.}$$

hence

$$A_{ij} = 0 \quad \text{if} \quad |i-j| > 1$$

$$A_{j-1,j} = A_{j,j+1} = -\frac{1}{h}$$

$$A_{j,j} = \frac{1}{h^2}\int_{x_{j-1}}^{x_{j+1}} dx = \frac{2}{h}$$

Hence

$$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \cdots & 0 \\ & & & & & \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Therefore, upto a scaling, the stiffness matrix in FEM equals the matrix that arises in a finite difference method..

To evaluate the load vector F_j:

$$F_j = \int_0^1 f(x)\,\varphi_j(x)\,dx$$

we need to use a quadrature rule to evaluate F_j

if we use midpoint rule.

$$F_j = \int_0^1 f(x)\,\varphi_j(x)\,dx = \int_{x_{j-1}}^{x_{j+1}} f(x)\,\varphi_j(x)\,dx$$

$$\approx 2h\,f(x_j) = 2h\,f_j.$$

7.4.3: Convergence analysis

let u_h be the FEM Solution (it satisfies (7.12))

define:
$$e_h := u - u_h, \quad e_h \in V, \text{ is the error function.}$$

we have by (7.10)
$$(u', v') = (f, v), \quad \forall v \in V_h \subset V$$

by FEM (7.12)
$$(u_h', v') = (f, v) \quad \forall v \in V_h$$

by the linearity of $(\cdot,\cdot)$, and by subtracting the above, we obtain,
$$((u-u_h)', v') \equiv 0, \quad \forall v \in V_h$$

by defn $\Longrightarrow$
$$(e_h', v') \equiv 0 \quad \text{— (7.16)}, \quad \forall v \in V_h$$

The above identity is termed as Galerkin orthogonality i.e. the error is orthogonal to the subspace V_h.

for any $v \in V_h$, define $w = u_h - v \in V_h$

then.
$$\|e_h\|^2_{H_0^1} = \int_0^1 |e_h'(x)|^2\, dx$$
$$= (e_h', e_h').$$
$$= (e_h', e_h') + (e_h', w') \quad \text{(Galerkin orthogonality (7.16))}$$
$$= (e_h', (e_h + w)') \quad \text{(linearity of } (\cdot,\cdot))$$
$$= (e_h', (u-v)') \quad \text{(defn. of } e_h)$$

$$\therefore \quad \|e_n\|^2_{H^1_0} = (e_n', (u-v)')$$

$$= \int_0^1 e_n'(x)\,(u'(x) - v'(x))\,dx$$

$$\le \left(\int_0^1 |e_n'|^2\,dx\right)^{1/2} \left(\int_0^1 |u'(x) - v'(x)|^2\,dx\right)^{1/2}$$

(Cauchy-Schwarz)

$$= \|e_n\|_{H^1_0}\,\|u-v\|_{H^1_0}$$

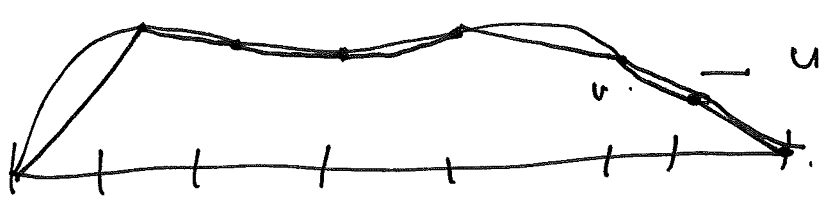
Hence. $\quad \|e_n\|_{H^1_0} \le \|u-v\|_{H^1_0}, \quad \forall\, v \in V_h.$

or $\quad \|u-u_n\|_{H^1_0} \le \|u-v\|_{H^1_0}, \quad \forall\, v \in V_h.$ —— (7.17)

In particular, we can choose v to be the piecewise linear interpolant of u i.e. on the given grid, v is a continuous piecewise linear function such that

$$u(x_i) = v(x_i), \quad \forall\, i$$

$$v(0) = v(1) = 0 \quad , \quad \text{see figure (7.4)}$$



Such a function v is denoted as $I_n u$,

from interpolation, we can calculate that

$$\cancel{\|u - I_n u\|_{H^1_0}} \le$$

$$|u(x) - I_h u(x)| \le \frac{h^2}{8} \max_{0 \le y \le 1} |u''(y)|$$

and $\quad |u'(x) - I_n u'(x)| \le C\,h \max\limits_{0 \le y \le 1} |u''(y)|$

∴, by squaring and integrating, we obtain

$$\|u - I_h u\|_{H_0^1} \leq Ch \|u''\|_{L^2} \quad \text{— (7.18)}$$

Hence by combining (7.17), and (7.18), we obtain

$$\|u - u_h\|_{H_0^1} \leq Ch \|u''\|_{L^2}$$

Furthermore by Poincare inequality,

$$\|u - u_h\|_{L^2} \leq Ch \|u''\|_{L^2}$$

Thus, FEM converges as $h \to 0$ and the rate of convergence in H_0^1 norm is 1 !!!