Musterlösung Lin. Alg. und Num.Math. D-BAUG Sommer 201

1.a)
$$\left[\begin{array}{ccc|c} a & 2 & 2a & 2b \\ 1 & a & 1 & b \\ 0 & 1 & 1 & b \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & a & 1 & b \\ 0 & 2-a^2 & \underbrace{2a-a}_{=a} & 2b-ab \\ 0 & 1 & 1 & b \end{array}\right]$$

b) $b=0$

$$\left[\begin{array}{ccc|c} 1 & a & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & (a-1)(a+2) & 0 \end{array}\right]$$

$a \in \{+1,-2\}$: Das LGS hat ∞-viele Lösungen $\to$ Falsch.

Gegenbeispiel: $a=+1$:
$$x = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$ ist nichttriviale Lösung

c) $b=1$

$a \in \mathbb{R}\setminus\{1,-2\}$:
$$\left.\begin{array}{ccc|c} 1 & a & 1 & \cancel{b}\,1 \\ 0 & 1 & 1 & \cancel{b}\,1 \\ 0 & 0 & (a-1)(a+2) & a\cancel{b}(a-1) \end{array}\right.$$

$\hookrightarrow z = \dfrac{a(a-1)}{(a-1)(a+2)} = \dfrac{a}{a+2}$

$\hookrightarrow y = 1 - \dfrac{a}{a+2} = \dfrac{2}{a+2}$

$\hookrightarrow x = 1 - \dfrac{a}{a+2} - \dfrac{2a}{a+2} = \dfrac{a+2-3a}{a+2} = \dfrac{-2a+2}{a+2} = \cancel{\dfrac{-2(a+1)}{a+2}}\ \dfrac{2(1-a)}{a+2}$

$$\Rightarrow \mathbb{L} = \left\{ \begin{pmatrix} 2(1-a) \\ 2 \\ a \end{pmatrix} \cdot \frac{1}{a+2} \right\}$$ $\Rightarrow$ genau eine Lösung bei $a \neq 1,-2$.

$a = 1$:

$$\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}$$

$\hookrightarrow z = s$

$\hookrightarrow y = 1 - s$

$\hookrightarrow x = 1 - s - (1 - s) = 0$

$$\Rightarrow \mathbb{L} = \left\{ \begin{pmatrix} 0 \\ 1-s \\ s \end{pmatrix} \middle| \, s \in \mathbb{R} \right\}$$

unendlich viele Lösungen bei $a = 1$

$a = -2$:

$$\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 6 \end{array} \quad \lightning \qquad \mathbb{L} = \emptyset$$

keine Lösung bei $a = -2$

2.

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & -1 & 2 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 8 & 0 & -4 \\ 0 & 8 & 0 \\ -4 & 0 & 8 \end{pmatrix}, \qquad A^T \cdot b = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}.$$

$$\begin{array}{ccc|c} 8 & 0 & -4 & -1 \\ 0 & 8 & 0 & 2 \\ -4 & 0 & 8 & 5 \end{array} \rightsquigarrow \begin{array}{ccc|c} 1 & 0 & -2 & -\frac{5}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 2 & 0 & -1 & -\frac{1}{4} \end{array}$$

$$\rightsquigarrow \begin{array}{ccc|c} 1 & 0 & -2 & -\frac{5}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 3 & -\frac{1}{4} + \frac{5}{2} = \frac{10-1}{4} = \frac{9}{4} \end{array}$$

$\hookrightarrow c = 3/4$

$\hookrightarrow b = 1/4$

$\hookrightarrow a = -\frac{5}{4} + 2 \cdot 3/4 = \frac{6-5}{4} = 1/4$

$$\Rightarrow \underline{a = 1/4, \; b = 1/4, \; c = 3/4}$$

3. a) $\det(A-2I) \overset{!}{=} 0$

$$\begin{vmatrix} 1-2 & -8 & 0 \\ 1 & -2-2 & 3 \\ 0 & 3 & s-2 \end{vmatrix} = \begin{vmatrix} -1 & -8 & 0 \\ 1 & -4 & 3 \\ 0 & 3 & s-2 \end{vmatrix} = 4(s-2)+0+0 - 0 - (-8)(s-2) - (-1)\cdot 9$$

$$= 12(s-2)+9 \overset{!}{=} 0 \Leftrightarrow s-2 = -\frac{9}{12} = -\frac{3}{4}$$

$$\Leftrightarrow s = \frac{8}{4} - \frac{3}{4} = \frac{5}{4}$$

b) $s=2$:

$$\begin{vmatrix} 1-\lambda & -8 & 0 \\ 1 & -2-\lambda & 3 \\ 0 & 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(-2-\lambda)(2-\lambda) + 0 + 0 - 0 - (-8)(2-\lambda) - 9\cdot(1-\lambda)$$

$$= [(-2+2\lambda-\lambda+\lambda^2)+8](2-\lambda) + 9\lambda - 9$$

$$= (\lambda^2+\lambda+6)(2-\lambda)+9\lambda-9 = 2\lambda^2+2\lambda+12-\lambda^3-\lambda^2-6\lambda+9\lambda-9$$

$$= -\lambda^3+\lambda^2+5\lambda+3 = 0$$

$$-\lambda^3+\lambda^2+5\lambda+3 = (\lambda+1)\cdot(-\lambda^2+2\lambda+3)$$

$$\begin{array}{l} -(-\lambda^3-\lambda^2) \\ \hline 2\lambda^2 \\ -(2\lambda^2+2\lambda) \\ \hline 3\lambda+3 \\ -(3\lambda+3) \\ \hline 0 \end{array}$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4+4\cdot 3}}{-2} = 1 \pm 2 = 3 \,/\, -1$$

$\lambda_1 = -1$, alg. Vielf.heit 2

$\lambda_2 = 3$, alg. Vielf.heit 1

$$|A-\lambda I| = -(\lambda+1)^2\cdot(\lambda-3)$$

$$\lambda_1: \left(\begin{array}{ccc|c} 2 & -8 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & -6 & -6 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right)$$

$\hookrightarrow z = s$

$\hookrightarrow y = -s$

$\hookrightarrow x = y - 3z = -s - 3s = -4s$

$\Rightarrow$ eigenspace $(-1) = \left\{ \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \cdot s \,\middle|\, s \in \mathbb{R} \right\}$

geom. Vielf.heit 1

$\lambda_2 = 3$:

$$\left(\begin{array}{ccc|c} -2 & -8 & 0 & 0 \\ 1 & -5 & 3 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right) \leadsto \left(\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ 0 & -18 & 6 & 0 \\ 0 & 3 & -1 & 0 \end{array}\right)$$

$\hookrightarrow z = s$

$\hookrightarrow y = \frac{1}{3}s$

$\hookrightarrow x = 5y - 3z = 5/3 \cdot s - 3 \cdot s = \frac{5-9}{3}s = -\frac{4}{3}s$

$\Rightarrow$ Eigenraum$(3) = \left\{ \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \cdot s \mid s \in \mathbb{R} \right\}$, geom. Vielf.heit $(3) = 1$

c)	alg. Vielf.heit	geom. Vielf.heit
-1	2	1
3	1	1

$\rightarrow$ Bei -1 ist alg. Vielf. $\neq$ geom. Vielf. $\Rightarrow$ nicht diagonalisierbar.

$\rightarrow$ Zwei Eigenvektoren, 3 Dimensionen $\Rightarrow$ keine Eigenbasis, nicht diagonalisierbar.

3.c) Nein, denn ER_1 und ER_2 spannen nicht ganz $\mathbb{R}^3$ auf.
geom. Vielf. (-1) < alg. Vielf. (-1)

4. a) $\dot{x}(t) = 0 \Leftrightarrow A \cdot x(t) = -b$

$$\left(\begin{array}{ccc|c} 2 & 0 & 2 & -4 \\ 0 & -1 & 0 & 1 \\ 2 & 0 & -1 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 0 & 2 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -3 & 3 \end{array}\right)$$

$x_2 = -1$

$x_3 = -1$

$x_1 + x_3 = -2$
$\Rightarrow x_1 = -2 + 1 = -1$

$$\Rightarrow \underline{x(t) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}}$$

b) $$0 = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & -1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(1+\lambda)^2 + 0 + 0 - 4 \cdot (-1-\lambda) - 0 - 0$$

$$= (2-\lambda)(1+\lambda)^2 + 4 \cdot (1+\lambda) = (1+\lambda) \cdot [(2-\lambda)(1+\lambda) + 4]$$

$$= (1+\lambda) \cdot [-\lambda^2 + \lambda + 6] = -(1+\lambda)(\lambda-3)(\lambda+2)$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+4 \cdot 6}}{2 \cdot (-1)} = \frac{-1 \pm 5}{-2} = \frac{-6}{-2} \,/\, \frac{4}{-2} = 3 \,/\, -2$$

Eigenwerte $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 3$

$\lambda_1 = -2$: $$\left(\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \underline{v_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}$$

$y = 0$ $\quad z = \alpha$
$2x + \alpha = 0$
$\Rightarrow x = -\alpha/2$

$\lambda_2 = -1$: $$\left(\begin{array}{ccc|c} 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \underline{v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$\lambda_3 = 3$: $$\left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -4 & 0 & 0 \\ 2 & 0 & -4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \underline{v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}}$$

$x = 2\alpha$ $\quad y = 0$ $\quad z = \alpha$

4. c)

$$U = [v_1\ v_2\ v_3] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$A \cdot U = U \cdot D \quad \Rightarrow \quad A = U D U^{-1}$$

$$\dot{x} = A \cdot x = U D U^{-1} x \quad | \; U^{-1} \cdot (\ldots)$$

$$\underbrace{U^{-1} \dot{x}(t)}_{= \dot{y}(t)} = D \underbrace{U^{-1} x(t)}_{= y(t)} \quad \Rightarrow \quad \dot{y} = D \cdot y$$

$$\begin{cases} \dot{y}_1 = -2 y_1 \\ \dot{y}_2 = -y_2 \\ \dot{y}_3 = 3 y_3 \end{cases} \quad \Rightarrow \quad \begin{cases} y_1(t) = C_1 \cdot e^{-2t} \\ y_2(t) = C_2 \, e^{-t} \\ y_3(t) = C_3 \, e^{3t} \end{cases}$$

$$y(t) = U^{-1} x(t) \Leftrightarrow \underline{x(t)} = U \cdot y(t) = \underline{\begin{pmatrix} C_1 e^{-2t} + 2 C_3 e^{3t} \\ C_2 e^{-t} \\ -2 C_1 e^{-2t} + C_3 e^{3t} \end{pmatrix}}$$

5. a) siehe Beiblatt

b) $F(x) = f(x) - g(x) = x + e^x - \frac{x}{x^2 - 4}$;

$x_0 = 0$

$x_1 = -1$

$$x_{n+1} = x_n - \frac{F(x_n)(x_n - x_{n-1})}{F(x_n) - F(x_{n-1})}$$

nötig

$x_2 = -0.5087883282 26848$

$x_3 = -0.485576565964323$

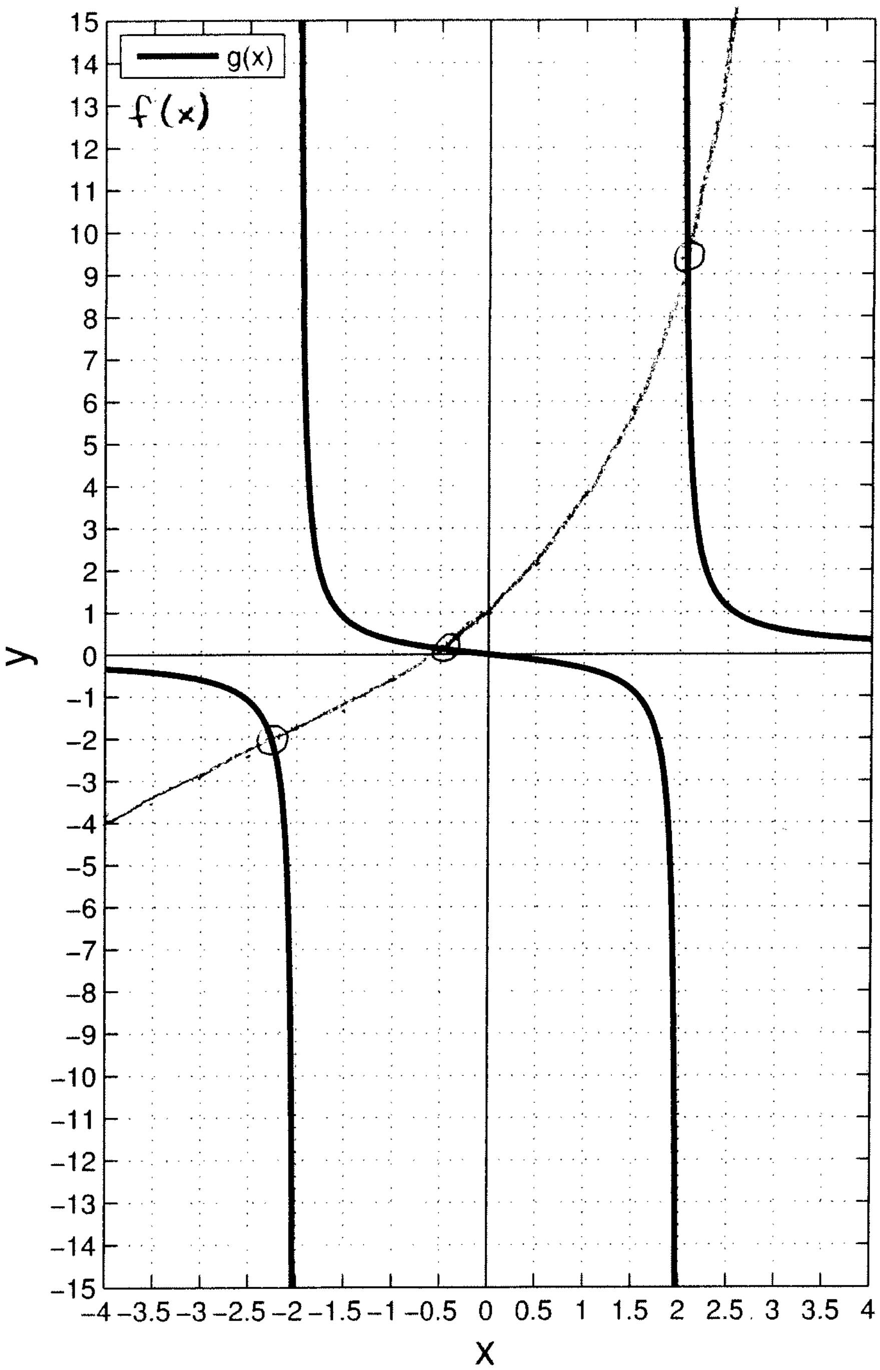
$x_4 = -0.4859785163198 16$

$x_5 = -0.4859776001404 39$

$x_6 = -0.4859776001027 96$

3 signifikante Stellen

Beiblatt zu Aufgabe 5. a)



Die Funktionen $f(x) = x + e^x$ und $g(x) = \frac{x}{x^2-4}$ haben 3 Schnittpunkte. Die x-Koordinaten dieser Schnittpunkte sind ungefähr: -2,25, -0.5, 2.1

6. a) $t_1 = \frac{1}{2}$, $t_2 = 1$:

$$\int_0^1 t^2 \cdot 1 \, dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}, \quad f(t_1) = 1, \quad f(t_2) = 1$$

$$\int_0^1 t^2 \cdot t \, dt = \frac{t^4}{4}\Big|_0^1 = \frac{1}{4}, \quad f(t_1) = \frac{1}{2}, \quad f(t_2) = 1$$

$$\begin{cases} w_1 \cdot 1 + w_2 \cdot 1 = \frac{1}{3} \\ w_1 \cdot \frac{1}{2} + w_2 \cdot 1 = \frac{1}{4} \end{cases} \Rightarrow \frac{1}{2} w_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \Rightarrow \underline{w_1 = \frac{1}{6}, \; w_2 = \frac{1}{6}}$$

b) $t_2 = 1$:

$$\int_0^1 t^2 \cdot 1 \, dt = \frac{1}{3}, \quad f(t_1) = 1, \quad f(t_2) = 1$$

$$\int_0^1 t^2 \cdot t \, dt = \frac{1}{4}, \quad f(t_1) = t_1, \quad f(t_2) = 1$$

$$\int_0^1 t^2 \cdot t^2 dt = \frac{t^5}{5}\Big|_0^1 = \frac{1}{5}, \quad f(t_1) = t_1^2, \quad f(t_2) = 1$$

$$w_2 = \frac{4(t_1-1)}{12(t_1-1)} \underset{\Uparrow}{-} \frac{-1}{12(t_1-1)} = \frac{4t_1 - 3}{12(t_1-1)}$$

$$\begin{cases} w_1 + w_2 = \frac{1}{3} \\ t_1 w_1 + w_2 = \frac{1}{4} \\ t_1^2 w_1 + w_2 = \frac{1}{5} \end{cases} \quad \Rightarrow w_2 = \frac{1}{3} - w_1$$

$$\Rightarrow t_1 w_1 + \frac{1}{3} - w_1 = \frac{1}{4} \Rightarrow w_1(t_1 - 1) = -\frac{1}{12} \Rightarrow w_1 = \frac{-1}{12(t_1-1)}$$

$$\Rightarrow t_1^2 \cdot \frac{-1}{12(t_1-1)} + \frac{4t_1 - 3}{12(t_1-1)} = \frac{1}{5} \qquad \Big| \cdot 12(t_1-1)$$

$$-t_1^2 + 4t_1 - 3 = \frac{12}{5}(t_1 - 1) \qquad \Big| \cdot 5$$

$$-5t_1^2 + 20t_1 - 15 - 12t_1 + 12 = -5t_1^2 + 8t_1 - 3 = 0$$

$$(t_1)_{1,2} = \frac{-8 \pm \sqrt{64 - 4 \cdot 3 \cdot 5}}{2 \cdot (-5)} = \frac{-8 \pm 2}{-10} = \frac{-10}{-10} \Big/ \frac{-6}{-10} = 1 \Big/ \frac{3}{5}$$

$$\Rightarrow \underline{t_1 = 3/5}$$

$$\underline{w_1} = \frac{-1}{12(-2/5)} = \underline{\frac{5}{24}}, \quad \underline{w_2} = \frac{8}{24} - \frac{5}{24} = \underline{\frac{3}{24}}$$

c)

$$\frac{|I_1 - \hat{I}_1|}{|I_1|} = 7.0432406703867 53 \cdot 10^{-5}$$

$$\frac{|I_2 - \hat{I}_2|}{|I_2|} = 8.73043486101 3003 \cdot 10^{-5}$$

$I_1 = 0,71823$

$I_2 = 0,16058$