Problem set – Week 7

Linear Algebra

1. Solve the following linear systems via elimination. Interpret your solutions geometrically.

$$(a)\quad \begin{aligned} x - 2y &= 2 \\ 3x + 5y &= 17 \end{aligned} \qquad (b)\quad \begin{aligned} x + y + z &= 0 \\ x - y + 2z &= 0 \end{aligned}$$

$$(c)\quad \begin{aligned} x + 4y + z &= 0 \\ 4x + 13y + 7z &= 0 \\ 7x + 22y + 13z &= 0 \end{aligned} \qquad (d)\quad \begin{aligned} x + 4y + z &= 0 \\ 4x + 13y + 7z &= 0 \\ 7x + 22y + 13z &= 1 \end{aligned}$$

2. Write the linear systems in question 1 in matrix form $Ax = b$.

3. Find the determinant of each of the matrices A in your answers to question 2 (a), (c) and (d). Are any of these matrices invertible? Find their inverse.

4. Compute the following matrix products, if possible.

$$(a)\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$(c)\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

5. Determine all eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

   Also, find the inverse of this matrix.

6. The eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue 1, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue 3. Write $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the basis of eigenvalues $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Write

Av in this basis as well.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $x \mapsto Ax$, and let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be $x \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} x$. What are the matrices for f and g in the basis $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?