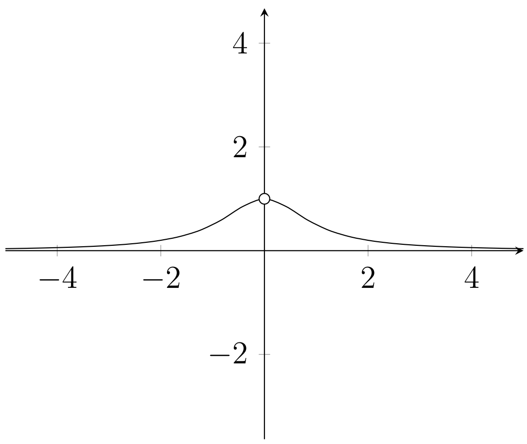
Solutions – Week 1

Limits and Differential Calculus

1. Consider the function $f : \mathbb{R}\setminus\{0\} \to \mathbb{R} :\ x \mapsto \frac{x}{x^3+x}$.

   (a) Draw a picture of graph (f).

   Solution : Since $x \neq 0$, we can make the simplification $f(x) = 1/(x^2+1)$. Note that this is a continuous and differentiable function attaining a maximum of 1 at $x = 0$, and approaching 0 as $x \to \infty$ or $x \to -\infty$.



   (b) What is $\text{im}(f)$? **Solution :** $(0, 1)$

   (c) What is $\lim_{x\to 0} f(x)$? **Solution :** 1

   (d) Where is f continuous? **Solution :** Everywhere in $\text{dom} f$ (i.e. $\mathbb{R}\setminus\{0\}$)

2. Give an example of a real function that is differentiable everywhere but its derivative is not continuous at zero. (Warning: this is a challenge question, you will need to think of quite an exotic function!)

   Solution : One example is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and $f(x) = x^2 \sin(1/x)$ for $x \neq 0$.

3. Compute the first derivative of

$$\text{(a) } x^3e^{-x^3} - x - 3, \qquad \text{(b) } \frac{\log(\sin^2(x))}{\cos(x)}, \qquad \text{(c) } \arctan(\sqrt{x}).$$

Solutions :

$$\text{(a) } 3x^2e^{-x^3}(1 - x^3) - 1, \quad \text{(b) } \frac{2}{\sin x} + \frac{\sin x}{\cos^2(x)} \log\sin^2(x), \quad \text{(c) } \frac{1}{2\sqrt{x}\,(1+x)}.$$

Compute the second derivative of

$$\text{(d) } \log(\log(x)).$$

Solution :

$$\frac{-1}{x^2\log(x)}\left(1 + \frac{1}{\log x}\right).$$

4. Suppose that a function f is continuous and differentiable in the interval $[0, 1]$. Suppose further that $f(0) = -1$ and $f'(x) \leq 2$ for all $x \in [0, 1]$. What is the largest possible value for $f(1)$?

   Solution : The mean value theorem says $f(1) - f(0) = f'(\xi)$ for some $\xi \in [0, 1]$. But since $f'(\xi) \leq 2$, this gives $f(1) \leq f(0) + 2 = 1$. Note that we get $f(1) = 1$ exactly if $f(x) = -1 + 2x$.

5. Let x, y be non-negative real numbers such that $x + y = 12$. What is the minimum possible value of x^2y ? For which values of x and y is this minimum attained?

   Solution : Since $y = 12 - x$, we have $x^2y = x^2(12 - x)$. Let $f(x) = x^2(12 - x)$. We need to find the minimum of $f(x)$ in the range $0 \leq x \leq 12$. This minimum can occur when $x = 0$, when $x = 12$ or when $f'(x) = 0$ (this last case occurs only when $x = 8$). Checking all three of these possibilities shows that the minimum $f(x) = 0$ is attained when $x = 0$ (thus $y = 12$), and when $x = 12$ (thus $y = 0$).