# Solutions – Week 10

## Linear Differential Equations

1. Solve the following differential equations:

$$(a)\ \frac{dy}{dx} = 5x - 3y/x,$$

$$(b)\ y'' + 7y' = 4y,$$

$$(c)\ y'' + 2y' = y,$$

$$(c)\ y'' + 4y = x.$$

**Solutions :**

(a) we first solve the homogeneous equation $dy/dx = -3y/x$:

$$\int dy/y = -3 \int dx/x$$
$$\log y = -3 \log x + c$$
$$y = ax^{-3}.$$

To find a particular solution we make the ansatz $y = bx^2$, because in this case $dy/dx$ and $3y/x$ and $5x$ will all be multiples of $x$. The differential equation with our ansatz gives $2bx = 5x - 3bx$, so $b = 1$. That is, the complete solution of this differential equation is

$$y = x^2 + ax^{-3}.$$

(b) The characteristic polynomial is $2\lambda^2 + 7\lambda - 4 = (\lambda + 4)(2\lambda - 1)$ which has solutions $\lambda = 1/2, 4$. So, the solution to the differential equation is $c_1 e^{x/2} + c_2 e^{-4x}$.

(c) The characteristic polynomial is $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$ which has a double root $\lambda = -1$. So, the solution to the differential equation is $c_1 e^{-x} + c_2 x e^{-x}$.

(d) The homogeneous equation has characteristic polynomial $\lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$, which has complex roots $\lambda = 2i, -2i$. The solution to the homogeneous equation is therefore $y = c_1 \cos(2x) + c_2 \sin(2x)$. For a particular solution, we make the ansatz $y = bx$ because then $y''$ is zero and $4y$ is comparable to $x$. We solve $b = 1/4$, so the solution to the differential equation is

$$y = c_1 \cos(2x) + c_2 \sin(2x) + x/4.$$

2. The following question appeared on last year's exam:

For which values of the parameter $\alpha \in \mathbb{R}$ does the differential equation

$$y'' + y' - 6y = -2\,\alpha\, e^x$$

have solutions that are bounded for $x \to \infty$? Determine all these solutions.

**Solution :** We first solve the homogeneous equation $y'' + y' - 6y = 0$. The characteristic polynomial is

$$\lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)\,.$$

The solution of the homogeneous equation is

$$y_h(x) = C_1 e^{2x} + C_2 e^{-3x}\,.$$

For the particular solution of the inhomogeneous equation we make the ansatz $y_p(x) = Be^x$. Then

$$\begin{aligned} y'' + y' - 6y &= Be^x + Be^x - 6Be^x \\ &= -4Be^x \overset{!}{=} -2\alpha e^x \end{aligned}$$

and $y_p(x) = \frac{\alpha}{2} e^x$. The general solution

$$y(x) = y_h(x) + y_p(x) = C_1 e^{2x} + C_2 e^{-3x} + \frac{\alpha}{2} e^x$$

is bounded if and only if $\alpha = 0$ and $C_1 = 0$. Therefore the general bounded solution is

$$y(x) = C_2 e^{-3x}\,.$$

3. The following question appeared on last year's exam:

   Determine the solution $y(x)$ of the differential equation

$$-xy' + y - 2xy^2 = 0$$

   that passes through the point $(1, 2)$.

   **Hint:** Substitute $z = \frac{x}{y}$.

   **Solution :** With $z = \frac{x}{y}$ ($y \not\equiv 0$ since $y(1) = 2 \neq 0$) we get $z' = \frac{y - xy'}{y^2}$ and with the differential equation

$$y^2(z' - 2x) = 0$$

   we solve

$$z' = 2x\,.$$

   This yields $z = x^2 + C$ and with $y = \frac{x}{z}$ we get the general solution

$$y = \frac{x}{x^2 + C}\,.$$

   The constant $C$ is determined by the initial condition $y(1) = 2$. The solution is

$$y = \frac{x}{x^2 - \frac{1}{2}}\,.$$

4. Solve

$$y^2 + y'(y^2x + 2xy - 1) = 0$$

   for $x$ as a function of $y$.
   **Hint:** use the ansatz $x = b/y^2$.

   **Solution :** Note that we are assuming $y \neq 0$, because this equation can never hold when $y = 0$. We have

$$y^2\frac{dx}{dy} + \left(y^2 + 2y\right)x - 1 = 0,$$
$$\frac{dx}{dy} = -\left(1 + \frac{2}{y}\right)x + \frac{1}{y^2}$$

   The homogeneous equation is

$$\frac{dx}{dy} = -\left(1 + \frac{2}{y}\right)x,$$

which we solve as follows:

$$\int \frac{dx}{x} = -\int \left(1 + \frac{2}{y}\right) dy$$
$$\log x = -y - 2\log y + c$$
$$x = ay^{-2}e^{-y}.$$

For the particular solution, the given ansatz gives $-2b/y^3 = -\left(1 + 2/y\right) b/y^2 - 1/y^2$, so $b = 1$ and the final solution is

$$x = ay^{-2}e^{-y} + y^{-2}.$$