Solutions – Week 2

Integration

1. Compute the following integrals.

(a) $\int (x^3 + 4x)\,\mathrm{d}x$ (b) $\int e^{-4x}\,\mathrm{d}x$ (c) $\int \sqrt{3x}\,\mathrm{d}x$

(d) $\int \cos(5x-2)\,\mathrm{d}x$ (e) $\int \cosh(5x-2)\,\mathrm{d}x$ (f) $\int (2x-5)^{-3/2}\,\mathrm{d}x$

(g) $\int_1^5 \frac{1}{x+3}\,\mathrm{d}x$ (h) $\int_9^{65} \frac{1}{3x-3}\,\mathrm{d}x$

Solutions (plus a constant for (a)-(f)) **:**

(a) $x^4/4 + 2x^2$ (b) $-e^{-4x}$ (c) $2x^{3/2}/\sqrt{3}$

(d) $\sin(5x-2)/5$ (e) $\sinh(5x-2)/5$ (f) $-1/\sqrt{2x-5}$

(g) $\ln 2$ (h) $\ln 2$

2. Compute the following integrals. (These are a bit more advanced!)

(a) $\int x \ln x\,\mathrm{d}x$ (b) $\int \frac{1}{x^2\sqrt{x^2+1}}\,\mathrm{d}x$ (c) $\int \frac{1}{x^2(x^2-1)}\,\mathrm{d}x$

(d) $\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$ (e) $\int \frac{dx}{x\ln x}\,\mathrm{d}x$ (f) $\int \frac{x\ dx}{(x^2-1)^{2/3}}\,\mathrm{d}x$

Solutions (all plus a constant) **:**

(a) $\frac{1}{4}x^2(2\ln|x|-1)$ (b) $-\sqrt{x^2+1}/x$ (c) $\frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\log|x+1|$

(d) $-2e^{-\sqrt{x}}$ (e) $\log\log x$ (f) $\frac{3}{2}\left(x^2-1\right)^{1/3}$

For (a), integrate by parts. For (b), consider the substitution $x = \tan u$ (so that $\sqrt{x^2+1} = 1/\cos u$). For (c), decompose into partial fractions. For the others, identify a function $g(x)$ such that the integrand is $g'(x)$ times some function of $g(x)$, so you can use the chain rule "in reverse" (or make the substitution $u = g(x)$, if you prefer). For example, in (f) we can take $g(x) = x^2$.

3. Figure 1 shows the graphs of the functions

$$f(x) = 4x^3 + 2x^2 - 5x - 2 \qquad \text{and} \qquad g(x) = 2x^2 - x - 2.$$

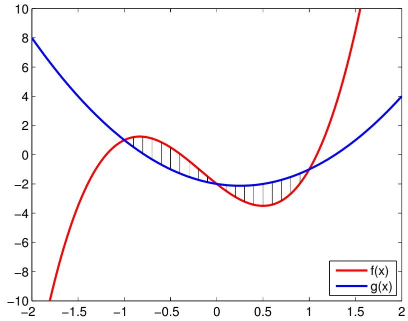

Figure 1: The functions $f(x)$ and $g(x)$ of Exercise 2

(a) Determine the x-coordinates $x_1 < x_2 < x_3$ of the points where the graphs intersect. **Solution :** $x_1 = -1$**,** $x_2 = 0$**,** $x_3 = 1$.

(b) Calculate the integral $\displaystyle\int_{x_1}^{x_3} \big(f(x) - g(x)\big)\,\mathrm{d}x$. **Solution :** 0

(c) Calculate the shaded area. **Solution :** Where f is above g, integrate $f - g$. Otherwise, integrate $g - f$. So the area is

$$\int_{-1}^{0} (f(x) - g(x))\,\mathrm{d}x + \int_{0}^{1} (g(x) - f(x))\,\mathrm{d}x = 2.$$

4. For which $x \in (0, 3\pi/2)$ is $f(x) = \int_x^{2x} \frac{\sin t}{t} dt$ a local maximum?

Solution : The fundamental theorem of calculus says that

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \frac{\sin t}{t} \mathrm{d}t = \frac{\sin x}{x},$$

and we can write

$$f(x) = \int_0^{2x} \frac{\sin t}{t} dt - \int_0^x \frac{\sin t}{t} dt,$$

so the chain rule says

$$f'(x) = 2\frac{\sin 2x}{2x} - \frac{\sin x}{x}.$$

This is zero when $\sin 2x = \sin x$. The double-angle formula gives $\sin 2x = 2\sin x \cos x$, so $\sin 2x = \sin x$ occurs when $\sin x = 0$ or when $\cos x = 1/2$. (in our range, this is when $x = \pi$ or $x = \pi/3$). These are the stationary points of $f(x)$, where local maxima could occur. Then, use the second-derivative test to see that only $x = \pi/3$ gives a local maximum.