Solutions – Week 3

Curves

1. Give a parameterization $r : \mathbb{R} \to \mathbb{R}^2$ of the circle defined by $x^2 + y^2 = 1/4$, in such a way that $|\dot{r}(t)| = 1$ for all t.

   Solution : We can parameterize the circle by $q : t \mapsto (\cos(t)/2, \sin(t)/2)$. This gives $|\dot{q}(t)| = 1/2$, so we can imagine a particle moving around the circle at speed $1/2$. We want to "speed" up this parameterization: the rotation should happen twice as fast. So we can take $r : t \mapsto (\cos(2t)/2, \sin(2t)/2)$.

2. Consider a circle of radius 1 centered at the point $(0, 1)$, and imagine that this circle is a wheel resting on the x-axis. Imagine that there is a red dot painted on the bottom of the wheel (at the point $(0, 0)$). Give a parameterization of the curve that is traced out by the red dot as the wheel rolls along the x-axis. If the wheel rolls at a constant speed, does the red dot also move at a constant speed?

   Solution : Imagine the wheel rolls at speed 1 to the right. The center of the wheel can be parameterized as $r_1 : t \mapsto (t, 1)$. So, the red dot is at position $(t + \cos\theta, 1 + \sin\theta)$, where θ is the angle of the red dot. The diameter of the wheel is $2\pi r$, so after $2\pi r$ time units it has made one complete rotation (that is, it has rotated $2\pi r$ units clockwise). Since θ changes at a constant (negative) speed and at time 0, $\theta = -\pi/2$, we can parameterize the movement of the red dot by

   $$r : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (t + \cos(-\pi/2 - t), 1 + \sin(-\pi/2 - t)) = (t + \sin t, 1 - \cos t).$$

   The speed of this curve is not constant.

3. In this problem we find the circumference of a quarter-circle three different ways.

   (a) Find the length of the curve $r : [0, 1] \to \mathbb{R}^2 : x \mapsto \left(x, \sqrt{1 - x^2}\right)$.

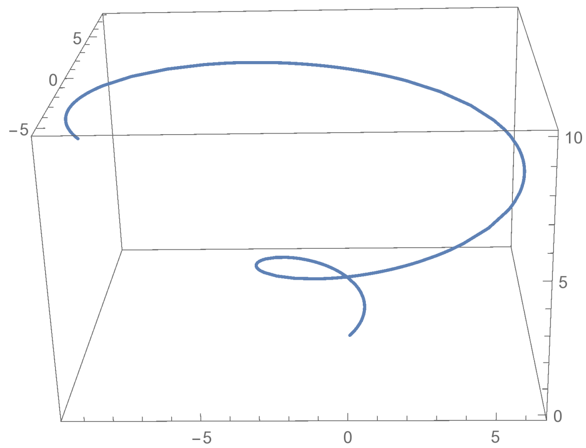
   (b) Find the length of the curve $r : [0, \pi/2] \to \mathbb{R}^2 : t \mapsto (\cos t, \sin t)$.

(c) Check that your answers agree with the formula for the circumference of a circle.

Solution : Each way should give $\pi/2$.

4. Find the length of the curve $r : [0,10] \to \mathbb{R}^2 : t \mapsto (t\cos t, t\sin t, t)$. Draw the image of this curve.

Solution : $5\sqrt{102} + \operatorname{arcsinh}\left(5\sqrt{2}\right)$.



5. Consider the force $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (x,1)$ and the curve $r : [0,1] \to \mathbb{R}^2 : t \mapsto (t, t^t)$. Compute the work of the force F along the curve r.

Solution : $\dot{r}(t) = \left(1, \frac{d}{dt}(t^t)\right)$ and $F(r(t)) = (t,1)$, so the work is $\int_0^1 \left(t + \frac{d}{dt}(t^t)\right) dt = t^2/2 + t^t|_0^1 = 1/2$ (note that we can take t^t to be 1 when $t = 0$, because $t^t = \exp(t\log t) \to \exp 0 = 1$ as $t \to 0$).