Solutions – Week 4

Work Along a Curve, Solids of Revolution

1. The following question is from last year's exam:
   Compute the work of the vector field $\vec{v}$

   $$(x, y, z) \longmapsto \vec{v}(x, y, z) := (y^2 z + 2x,\ 2xyz,\ xy^2)$$

   along the line segment that goes from $P_1 := (1, 0, 1)$ to $P_2 := (0, 1, 1)$.

   Solution :
   We parameterize the path from $P_1 := (1, 0, 1)$ to $P_2 := (0, 1, 1)$:

   $$\vec{r}(t) := (1 - t, t, 1)\,, \quad t \in [0, 1]\,.$$

   Then

   $$\dot{\vec{r}}(t) := (-1, 1, 0)\,.$$

   The vector field is

   $$\begin{aligned} \vec{v}\big(x(t), y(t), z(t)\big) &= \vec{v}\big(1 - t, t, 1\big) \\ &= (t^2 - 2t + 2, -2t^2 + 2t, -t^3 + t^2)\,. \end{aligned}$$

   The work of the vector field is

   $$\begin{aligned} A &= \int_0^1 \vec{v} \cdot \dot{\vec{r}} dt \\ &= \int_0^1 (t^2 - 2t + 2, -2t^2 + 2t, -t^3 + t^2) \cdot (-1, 1, 0) dt \\ &= \int_0^1 (-t^2 + 2t - 2 - 2t^2 + 2t)\, dt = \int_0^1 (-3t^2 + 4t - 2)\, dt \\ &= -t^3 + 2t^2 - 2t\Big|_0^1 = -1\,. \end{aligned}$$

2. The following question is from last year's exam:
   Compute the volume of the solid of revolution that is obtained by the revolution of the curve

   $$y = \frac{1}{\sqrt{1 - x^2}}\,, \quad -\frac{1}{2} \leq x \leq \frac{1}{2}$$

round the x-axis.

Solution :
The volume is

$$V = \int_{-1/2}^{1/2} \pi \left(\frac{1}{\sqrt{1-x^2}} \right)^2 dx = \pi \int_{-1/2}^{1/2} \frac{1}{1-x^2}\, dx\,.$$

The partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x} \right)$$

yields

$$\begin{aligned} V &= \frac{\pi}{2} \int_{-1/2}^{1/2} \left(\frac{1}{1-x} + \frac{1}{1+x} \right) dx \\ &= \frac{\pi}{2} \big(-\ln|1-x| + \ln|1+x| \big) \Big|_{-1/2}^{1/2} \\ &= \frac{\pi}{2} \big(-\ln|1-1/2| + \ln|1+1/2| + \ln|1+1/2| - \ln|1-1/2| \big) \\ &= \frac{\pi}{2} \cdot 2 \cdot \ln 3 = \pi \ln 3\,. \end{aligned}$$

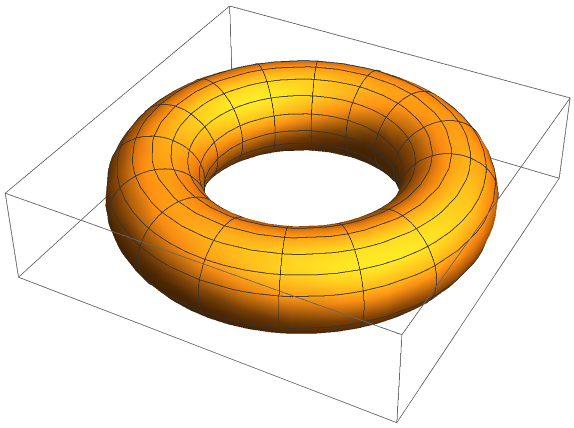
3. Consider the area enclosed by the curve

$$y = \frac{1}{\sqrt{1-x^2}}$$

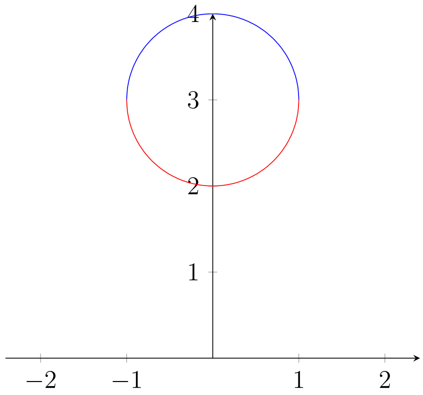
and the line $y = 2/\sqrt{3}$. Compute the volume obtained by revolving this area around the y-axis.

Solution :
We have $x = \sqrt{1-y^{-2}}$, so the volume is $\pi \int_1^{2/\sqrt{3}} (1 + y^{-2})\, dy = \pi \left(y + y^{-1} \big|_1^{2/\sqrt{3}} \right) = \pi \left(\frac{7}{2\sqrt{3}} - 2 \right)$.

4. Compute the volume of a *torus* (pictured), where the radial cross sections are circles with radius 1, and the centers of these circles trace out a circle of radius 3. (So, the hole in the middle has a radius of 2, and the entire object has a radius of 4.



Solution : In the picture below, let V_b the the volume of the convex solid obtained by rotating the blue curve $y = 3 + \sqrt{1-x^2}$ around the x axis. Similarly let V_r be the volume obtained by rotating the red curve around the x axis. We then subtract these volumes to get the volume of the torus.



This gives

$$V = \pi \int_{-1}^{1} \left(3 + \sqrt{1-x^2}\right)^2 dx - \pi \int_{-1}^{1} \left(3 - \sqrt{1-x^2}\right)^2 dx = 12\pi \int_{-1}^{1} \sqrt{1-x^2} dx = 6\pi^2.$$