Solutions – Week 6

Integration (review)

1. Consider the parabolic segment $y = x^2$ between $x = 1$ and $x = 2$, and the force $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (xy, x)$. Compute the work of the force F along the parabolic segment.

   Solution : This parabolic segment can be parameterized as $r : [1, 2] \to \mathbb{R}^2 : t \mapsto (t, t^2)$, and we can compute $\dot{r}(t) = (1, 2t)$. Along r, the work at time t is $F(r(t)) = (tt^2, t) = (t^3, t)$. The work is therefore

$$\begin{aligned}\int_1^2 (1, 2t) \cdot \left(t^3, t\right) dt &= \int_1^2 \left(1t^3 + 2tt\right) dt \\ &= \int_1^2 \left(t^3 + 2t^2\right) dt \\ &= \left(t^4/4 + 2t^3/3\right) |_1^2 \\ &= 4 + 16/3 - 1/4 - 2/3 = 101/12.\end{aligned}$$

2. Consider the line segment from $(0, 1)$ to $(1, 2)$. What is the surface area of the rotation of this line segment around the x-axis?

   Solution : We can parameterize this curve by $r : [0, 1] \to \mathbb{R}^2 : t \mapsto (0, 1) + t((1, 2) - (0, 1)) = (t, 1 + t)$, which satisfies $\dot{r}(t) = (1, 1)$ and $|\dot{r}(t)| = \sqrt{2}$. The surface area is then

$$2\pi \int_0^1 y\,|\dot{r}(t)|\,dt = 2\pi \int_0^1 \sqrt{2}\,(1 + t) = 2\sqrt{2}\pi \left(t + t^2/2\right) |_0^1 = 3\sqrt{2}\pi.$$

3. Consider the curve $t :\mapsto (t^2, t^3)$ from $t = 1$ to $t = 3$. What is the length of this curve?

   Solution : The speed of the curve is $\dot{r}(t) = (2t, 3t^2)$ with

$$|\dot{r}(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}.$$

   Then, the length is

$$\int_1^3 |\dot{r}(t)|\,dt = \int_1^3 t\sqrt{4 + 9t^2} dt = \left(4 + 9t^2\right)^{3/2} /27|_1^3 = \left(85^{3/2} - 13^{3/2}\right) /27.$$

4. Consider the curve $t \mapsto (\cos t, \sin(2t))$ from $t = 0$ to $t = \pi/2$. Rotate this curve around the x-axis. What is the resulting volume?

Solution : With $x = \cos t$, note that $dx/dt = -\sin t$ so the volume is

$$\begin{aligned}
\pi \int_0^{\pi/2} y^2 dx &= \pi \int_0^{\pi/2} \sin^2(2t) \sin t\, dt = \pi \int_0^{\pi/2} (2\cos t \sin t)^2 \sin t\, dt \\
&= 4\pi \int_0^{\pi/2} \cos t \left(1 - \cos^2 t\right) \sin t\, dt = 4\pi \int_0^{\pi/2} \left(\cos t \sin t + \sin t \cos^3 t\right) dt \\
&= 4\pi \left(\sin t - \cos^4 t/4\right) |_0^{\pi/2} = 5\pi.
\end{aligned}$$