Solutions – Week 7

Linear Algebra

1. Solve the following linear systems via elimination. Interpret your solutions geometrically.

$$(a)\ \begin{array}{rcrcr} x & - & 2y & = & 2 \\ 3x & + & 5y & = & 17 \end{array} \qquad (b)\ \begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ x & - & y & + & 2z & = & 0 \end{array}$$

$$(c)\ \begin{array}{rcrcrcr} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 0 \end{array} \qquad (d)\ \begin{array}{rcrcrcr} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 1 \end{array}$$

Solutions :

(a) Subtracting three times the first row from the second, we obtain $11y = 11$. Substituting $y = 1$ into the first row, $x = 4$. The point $(4, 1)$ is the intersection of the lines $\{y = \frac{x}{2} - 1\}$ and $\{y = -\frac{3}{5}x + \frac{17}{5}\}$.

(b) Subtracting the second row from the first gives $2y - z = 0$. Let $y = t$ be arbitrary. Substituting $z = 2t$ into the first row gives $x = 3t$, so the solution of the system is $(x, y, z) \in \{t\,(3, 1, 2) : t \in \mathbb{R}\}$. Geometrically, this line is the intersection of the two planes $x+y+z = 0$ and $x-y+2z = 0$.

(c) By elimination, we reduce the system to

$$\begin{array}{rcrcrcr} x & + & 4y & + & z & = & 0 \\ & - & 3y & + & 3z & = & 0 \\ & - & 6y & + & 6z & = & 0 \end{array}$$

and further to

$$\begin{array}{rcrcrcr} x & + & 5y & & & = & 0 \\ & - & y & + & z & = & 0. \end{array}$$

Set $y = t$ arbitrarily. Then $z = t$ and $x = -5t$. The set of solutions is the line $\{t(-5, 1, 1) : t \in \mathbb{R}\}$, which is the intersection of the three planes $x + 4y + z = 0$, $4x + 13y + 7z = 0$ and $7x + 22y + 13z = 0$.

(d) The system reduces to

$$\begin{array}{rcrcrcl} x & + & 4y & + & z & = & 0 \\ & - & 3y & + & 3z & = & 0 \\ & - & 6y & + & 6z & = & 1. \end{array}$$

Comparing the last two rows, we can see that the system is inconsistent; it can have no solutions, because $1 \neq 0$. This means the three planes $x+4y+z=0$, $4x+13y+7z=0$ and $7x+22y+13z=1$ have no point in common.

2. Write the linear systems in question 1 in matrix form $Ax = b$

Solutions :

$(a)\ \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 17 \end{pmatrix}$ $\quad (b)\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(c)\ \begin{pmatrix} 1 & 4 & 1 \\ 4 & 13 & 7 \\ 7 & 22 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad (d)\ \begin{pmatrix} 1 & 4 & 1 \\ 4 & 13 & 7 \\ 7 & 22 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

3. Find the determinant of each of the matrices A in your answers to question 2 (a), (b) and (c). Are any of these matrices invertible? Find their inverse.

Solutions : In (a), A has determinant 11 and inverse $\frac{1}{11}\begin{pmatrix} 5 & 2 \\ -3 & 1 \end{pmatrix}$. In (c) and (d) A has determinant 0, and is not invertible.

4. Compute the following matrix products, if possible.

$(a)\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad (b)\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$(c)\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Solutions : the matrices in (b) do not have appropriate dimensions so cannot be multiplied.

$(a)\ \begin{pmatrix} 1+3 & 2+4 \\ 0+3 & 0+4 \\ 1+0 & 2+0 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 3 & 4 \\ 1 & 2 \end{pmatrix}$ $\quad (c)\ (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

5. Determine all eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

Also, find the inverse of this matrix.

Solutions : Denote the matrix by A. We have

$$\det (A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & 2-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} = -\lambda (2-\lambda)(1-\lambda) - (1-\lambda) - 4(2-\lambda) = -\lambda^3 + 3\lambda^2 + 3\lambda -$$

so $A = \lambda I$ has nontrivial solutions only when $\lambda \in \{-\sqrt{3}, \sqrt{3}, 3\}$. These are the 3 eigenvalues. To find an eigenvector for $\lambda = -\sqrt{3}$, we solve the system

$$\left(A + \sqrt{3} I\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

By elimination this gives $(x, y, z) = \left\{ t \begin{pmatrix} 1 \\ \sqrt{3}-2 \\ 1-\sqrt{3} \end{pmatrix} \right\}$, so an eigenvector is $\begin{pmatrix} 1 \\ \sqrt{3}-2 \\ 1-\sqrt{3} \end{pmatrix}$. An eigenvector for $\lambda = \sqrt{3}$ is $\begin{pmatrix} 1 \\ -\sqrt{3}-2 \\ 1+\sqrt{3} \end{pmatrix}$, and an eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

6. The eigenvectors of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue 1, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with eigenvalue 3. Write $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the basis of eigenvalues $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Write Av in this basis as well.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function $x \mapsto Ax$, and let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be $x \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} x$. What are the matrices for f and g in the basis $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

Solutions : We have

$$v = \frac{3}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so in the given basis v has representation $(-1/2, 3/2)$. Then Av has representation $(-1/2 \times 1,\ 3/2 \times 3) = (-1/2,\ 9/2)$. The matrix for f in the new basis is $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

The change-of-basis matrix is $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ with inverse $T^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ so in the new basis g has matrix

$$T^{-1}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} T = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 3 \end{pmatrix}.$$