Solutions – Week 9

Extrema, Differential Equations

1. Consider the function $f : (x, y) \mapsto x^3 - x + y^3 - y$. Find its stationary points and use the Hessian matrix to determine whether each is a local minimum, maximum or saddle point.

   Solution : We have $\nabla f = (3x^2 - 1, 3y^2 - 1)$ so there are stationary points at $\left(1/\sqrt{3}, 1/\sqrt{3}\right)$, $\left(1/\sqrt{3}, -1/\sqrt{3}\right)$, $\left(-1/\sqrt{3}, 1/\sqrt{3}\right)$ and $\left(-1/\sqrt{3}, -1/\sqrt{3}\right)$. Then $H = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$ which has eigenvalues $6x, 6y$, so the stationary points are a local minimum, saddle point, saddle point and local maximum respectively.

2. Solve the following differential equations, both generally and with the condition $y(0) = 1$.

$$(a)\ \frac{dy}{dx} = 3e^y x^2$$

   Solution :

$$\int e^{-y} dy = \int 3x^2 dx,$$
$$-e^{-y} = x^3 + c,$$
$$y = -\log\left(-x^3 - c\right).$$

   To satisfy the initial condition, set $c = -1/e$.

$$(b)\ \frac{dy}{dx} = \frac{2 - \sin(x + 2y)}{2\sin(x + 2y)}.$$

   Solution : Set $u = x + 2y$ so that $du/dx = 1 + 2dy/dx$ and we can instead solve

$$\frac{du}{dx} - 1 = \frac{2 - \sin u}{\sin u}.$$

   We have

$$\int \frac{du}{1 + \frac{2 - \sin u}{\sin u}} = \frac{1}{2} \int \sin u \, du = \int dx,$$
$$-\frac{1}{2} \cos u = x + c.$$

One family of solutions is therefore $u = \arccos(c' - 2x)$, and most generally the solution is $u = 2k\pi + s \arccos(c' - 2x)$ for some $k \in \mathbb{Z}$ and $s \in \{-, 1, 1\}$. (Note that arccos is not a true inverse of cos.) Therefore $y = 2k\pi + s \arccos(c' - 2x)/2 - x/2$. To satisfy the initial condition, we must have $k = 0$ and $s = 1$, and we can take $c = \cos 2$.

$$(c)\ \frac{dy}{dx} = \left(\frac{y}{x+1}\right)^2 + \frac{y}{x+1}.$$

Solution : Make the substitution $u = y/(x+1)$, so that $dy/dx = (x+1)\, du/dx + u$ and

$$\begin{aligned}(x+1)\frac{du}{dx} + u &= u^2 + u,\\ \int \frac{du}{u^2} &= \int \frac{dx}{x+1},\\ -u^{-1} &= \log(x+1) + c.\end{aligned}$$

So, $u = -1/(\log(x+1) + c)$ and $y = -(x+1)/(\log(x+1) + c)$. To satisfy the initial condition, we must have $c = -1$.