Absolute values, valuations and completion

F. Crivelli (flcrivel@student.ethz.ch)
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Introduction

During this talk I’ll introduce the basic definitions and some results about valuations, absolute values of fields and completions. These notions will give two basic examples: the rational function field $\mathbb{F}_q(T)$ and the field $\mathbb{Q}_p$ of the $p$-adic numbers.

1 Absolute values and valuations

All along this section, $K$ denote a field.

1.1 Absolute values

We begin with a well-known

Definition 1. An absolute value of $K$ is a function

$$| | : K \to \mathbb{R}$$

satisfying these properties, $\forall x, y \in K$:

1. $|x| = 0 \iff x = 0$,
2. $|x| \geq 0$,
3. $|xy| = |x||y|$,
4. $|x + y| \leq |x| + |y|$. (Triangle inequality)

Note that if we set $|x| = 1$ for all $0 \neq x \in K$ and $|0| = 0$, we have an absolute value on $K$, called the trivial absolute value. From now on, when we speak about an absolute value $| |$, we assume that $| |$ is non-trivial. Moreover, if we define $d : K \times K \to \mathbb{R}$ by $d(x, y) = |x - y|$, $x, y \in K$, $d$ is a metric on $K$ and we have a topological structure on $K$. We have also some basic properties that we can deduce directly from the definition of an absolute value.
Lemma 1. Let $\lvert \cdot \rvert$ be an absolute value on $K$. We have

1. $\lvert 1 \rvert = 1$,
2. $\lvert \zeta \rvert = 1$, for all $\zeta \in K$ with $\zeta^d = 1$ for some $0 \neq d \in \mathbb{N}$ ($\zeta$ is a root of unity),
3. $\lvert x^{-1} \rvert = \lvert x \rvert^{-1}$,
4. $\lvert |x| - |y| \rvert \leq |x - y|$.

Proof:

1. $\lvert 1 \rvert^2 = |1^2| = |1| \Rightarrow \lvert 1 \rvert = 1$.
2. $\lvert \zeta^d \rvert = |\zeta^d| = |1| = 1 \Rightarrow \lvert \zeta \rvert = 1$.
3. $1 = \lvert xx^{-1} \rvert = \lvert x \rvert \lvert x^{-1} \rvert \Rightarrow \lvert x^{-1} \rvert = \lvert x \rvert^{-1}$,
4. Set $a := x - y$, $b := y$. We have $|a + b| \leq |a| + |b|$, therefore $|x| - |y| = |a + b| - |b| \leq |a| = |x - y|$.

□

Another definition about absolute values.

Definition 2. Two absolute values on $K$ are **equivalent** if they define the same topology on $K$.

The next theorem and its corollary give us other conditions to verify if two absolute values are equivalent.

Theorem 1. Let $\lvert \cdot \rvert_1$ and $\lvert \cdot \rvert_2$ be two absolute values on $K$. They are equivalent if, and only if, there exists $s \geq 0$ real such that

$$\lvert x \rvert_1 = \lvert x \rvert_2^s, \forall x \in K.$$ 

Proof: If we have $\lvert x \rvert_1 = \lvert x \rvert_2^s$, $s > 0$, clearly the two absolute values are equivalent since they define the same open sets.

If $\lvert \cdot \rvert_1$ and $\lvert \cdot \rvert_2$ are equivalent, a series converging to 0 with respect to $\lvert \cdot \rvert_1$ will converge to 0 with respect to $\lvert \cdot \rvert_2$. Moreover, for all $K$ field and $\lvert \cdot \rvert$ on $K$, the inequality $|x| < 1$ is equivalent to saying that the sequence ${x^n}_{n \in \mathbb{N}}$ converges to 0. Therefore if $\lvert \cdot \rvert_1$ and $\lvert \cdot \rvert_2$ are equivalent, we have

$$\lvert x \rvert_1 < 1 \Leftrightarrow \lvert x \rvert_2 < 1.$$
Let $y \in K$ be an element such that $|y|_1 > 1$ and let $x \in K$, $x \neq 0$. Then there exists $\alpha \in \mathbb{R}$ such that $|x|_1 = |y|_1^\alpha$. Let $\left\{ \frac{m_i}{n_i} \right\}_{i \in \mathbb{N}}$ be a sequence of rational numbers ($m_i \in \mathbb{Z}$, $n_i \in \mathbb{N}^*$) converging to $\alpha$ from above. Then $|x|_1 = |y|_1^{\frac{m_i}{n_i}}$ and hence

$$\frac{|x^{m_i}}{y^{m_i}} |_1 < 1 \Rightarrow \frac{|x|}{y^{m_i}} |_2 < 1.$$ 

This gives $|x|_2 < |y|_2^{\frac{m_i}{n_i}}$ and consequently $|x|_2 \leq |y|_2^{\alpha}$. Taking a sequence $\left\{ \frac{m_i}{n_i} \right\}_{i \in \mathbb{N}}$ converging to $\alpha$ from below will give $|x|_2 \geq |y|_2^{\alpha}$ and therefore $|x|_2 = |y|_2^{\alpha}$. So, for all $0 \neq x \in K$, we have

$$\frac{\log |x|_1}{\log |x|_2} = \frac{\log |y|_1}{\log |y|_2} =: s$$

and hence $|x|_1 = |x|_2^s$. Finally, $|y|_1 > 1$ implies $|y|_2 > 1$ and so $s > 0$.

□

**Corollary.** Two absolute values $| \cdot |_1$ and $| \cdot |_2$ on $K$ are equivalent if, and only if,

$$|x|_1 < 1 \iff |x|_2 < 1.$$ 

We continue with this important

**Definition 3.** An absolute value is called **non-Archimedean** if we have

$$|x + y| \leq \max\{|x|, |y|\}, \forall x, y \in K.$$ 

Otherwise the absolute value is called **Archimedean**.

Note that if $| \cdot |$ is non-Archimedean, for $x, y \in K$ we have

$$|x| \neq |y| \implies |x + y| = \max\{|x|, |y|\}.$$ 

Indeed: w.l.o.g we can assume $|x| > |y|$, then obviously $|x + y| \leq \max\{|x|, |y|\} = |x|$. On the other side, $|x| = |x - y + y| \leq \max\{|x - y|, |y|\}$. Assume that $\max\{|x + y|, |y|\} = |y|$, then $|x| \leq |y| < |x|$. This contradicts the hypothesis $|x| > |y|$, hence $\max\{|x + y|, |y|\} = |x + y|$ and so $|x| \leq |x + y|$.

**1.2 Valuations**

We introduce the symbol $\infty$ with the convention that for all $a \in \mathbb{R}$ we have $a < \infty$, $a + \infty = \infty$ and $\infty + \infty = \infty$. As for absolute values, we start with a basic

**Definition 4.** A valuation on $K$ is a function

$$v : K \to \mathbb{R} \cup \{\infty\}$$

satisfying these properties, for all $x, y \in K$:
1. \( v(x) = \infty \iff x = 0 \),
2. \( v(xy) = v(x) + v(y) \),
3. \( v(x + y) \geq \min\{v(x), v(y)\} \).

Note that if we set \( v(x) = 0 \) for all \( 0 \neq x \in K \) and \( v(0) = \infty \), we have a valuation on \( K \), called the trivial valuation. From now on, when we speak about a valuation \( v \), we assume that \( v \) is non-trivial.

Now, a lemma with some basic properties induced by the definition of valuation

**Lemma 2.** Let \( v \) be a valuation on \( K \). We have

1. \( v(1) = 0 \),
2. \( v(\zeta) = 0 \), for all \( \zeta \in K \) root of unity,
3. \( v(x^{-1}) = -v(x), \forall x \in K^* \),
4. if \( x, y \in K \) and \( v(x) \neq v(y) \), \( v(x + y) = \min\{v(x), v(y)\} \).

**Proof:**

1. \( v(1) = v(1^2) = v(1) + v(1) \Rightarrow v(1) = 0 \).
2. Let \( \zeta \in K \) with \( \zeta^d = 1 \) for some \( 0 \neq d \in \mathbb{N} \). We have \( dv(\zeta) = v(\zeta^d) = v(1) = 0 \), therefore \( v(\zeta) = 0 \).
3. \( 0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1}) \Rightarrow v(x^{-1}) = -v(x) \).
4. W.l.o.g. we can assume that \( v(x) > v(y) \) and so \( y \neq 0 \). If \( x = 0 \), obvious. Assume \( x \neq 0 \). Then \( v(x + y) \geq \min\{v(x), v(y)\} = v(y) \). We have also \( v(y) = v(y + x - x) \geq \min\{v(x + y), v(x)\} \). Assume that \( \min\{v(x + y), v(x)\} = v(x) \), then \( v(x) > v(y) \geq v(x) \) and we get a contradiction. So, \( \min\{v(x + y), v(x)\} = v(x + y) \) and finally \( v(y) \geq v(x + y) \).

}\[\square\]

Some further terminology regarding valuations.

**Definition 5.** A valuation \( v \) on \( K \) is called **discrete** if \( v(K^*) = s\mathbb{Z} \), for a real \( s > 0 \). Moreover, \( v \) is **normalized** if \( s = 1 \).

We introduce now the equivalence between valuations.

**Definition 6.** Two valuations \( v_1 \) and \( v_2 \) on \( K \) are **equivalent** if there exists a real \( s > 0 \) such that \( v_1 = sv_2 \).

Note that if we have a discrete valuation on \( K \) with \( v(K^*) = s\mathbb{Z} \), dividing it by \( s \) we obtain an equivalent normalized valuation.
1.3 Relations between non-Archimedean absolute values and valuations

The following theorem provides a relation between the non-Archimedean absolute values and the valuations on $K$.

**Theorem 2.** Let $| \cdot |$ be a non-Archimedean absolute value on $K$ and $s \in \mathbb{R}$, $s > 0$, then the function

$$v_s : K \rightarrow \mathbb{R} \cup \{\infty\}$$

$$x \mapsto \begin{cases} -s \log |x| & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

is a valuation on $K$. Furthermore, if $s, s' \in \mathbb{R}$, $s, s' > 0$ and $s \neq s'$, $v_s$ is equivalent to $v_{s'}$. Conversely, if $v$ is a valuation on $K$ and $q \in \mathbb{R}$, $q > 1$, the function

$$| \cdot |_q : K \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} q^{-v(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is an absolute value on $K$. Besides, if $q, q' \in \mathbb{R}$, $q, q' > 1$ and $q \neq q'$, $| \cdot |_q$ is equivalent to $| \cdot |_{q'}$.

**Proof:** We just need to check the definitions of an absolute value and a valuation. We start with $v_s$. Clearly we have that $v_s(x) = \infty$ if, and only if, $x = 0$. Let $x, y \in K$, if $x = 0$ or $y = 0$, $xy = 0$ and $\infty = v_s(xy) = v(x) + v(y) = \infty$. Assume $x, y \neq 0$, then

$$v_s(xy) = -s \log |xy| = -s \log (|x| \cdot |y|) = -s \log |x| - s \log |y| = v_s(x) + v_s(y).$$

Let again be $x, y \in K$. If $x = y = 0$, then $\infty = v_s(x + y) = \min \{v_s(x), v_s(y)\} = \infty$.

If $x = 0$, $y \neq 0$ (or $y = 0$, $x \neq 0$), $v_s(x + y) = v_s(y) = \min \{v_s(x), v_s(y)\}$ (or $v_s(x + y) = v_s(x) = \min \{v_s(x), v_s(y)\}$). Assume now $x, y \neq 0$. We have

$$v_s(x + y) = -s \log |x + y| \geq -s \log (\max \{|x|, |y|\}) = -s \log |x| - s \log |y| = \min \{v_s(x), v_s(y)\}.$$

Therefore $v_s$ is a valuation on $K$. Assume now $s, s' > 0$, $s \neq s'$. For all $0 \neq x \in K$, we have

$$v_s(x) = -s \log |x| = \left(\frac{s}{s'}\right) (-s' \log |x|) = \frac{s}{s'} v_{s'}(x).$$

This means that $v_s$ and $v_{s'}$ are equivalent. We continue with $| \cdot |_q$. Clearly we have that $|x| = 0$ if, and only if, $x = 0$ and since $q > 1 > 0$ that $|x|_q \geq 0$ for all $x \in K$.

Let $x, y \in K$, if $x = 0$ or $y = 0$, $xy = 0$ and $0 = |xy|_q = |x|_q |y|_q = 0$. Assume $x, y \neq 0$, then

$$|xy|_q = q^{-v(xy)} = q^{-v(x)} q^{-v(y)} = q^{-v(x)} q^{-v(y)} = |x|_q |y|_q.$$
Since \(\max\left\{ |x|_q, |y|_q \right\} \leq |x|_q + |y|_q\), it suffices to show that
\[|x + y|_q \leq \max\left\{ |x|_q, |y|_q \right\} .\]
Let again be \(x, y \in K\). If \(x = y = 0\),
\[0 = |x + y|_q = \max\left\{ |x|_q, |y|_q \right\} = 0 .\]
If \(x = 0, y \neq 0 \) (or \(y = 0, x \neq 0\)),
\[|x + y|_q = |y|_q = \max\left\{ |x|_q, |y|_q \right\} \text{ (or } |x + y|_q = |x|_q = \max\left\{ |x|_q, |y|_q \right\}) .\]
Assume now \(x, y \neq 0\). We have
\[
|x + y|_q = q^{-v(x+y)} \\
\leq q^{-\min\{v(x),v(y)\}} \\
= \max\left\{ q^{-v(x)}, q^{-v(y)} \right\} \\
= \max\left\{ |x|_q, |y|_q \right\} .
\]
Therefore, \(| |_q\) is a non-Archimedean absolute value on \(K\). Assume now \(q, q' > 1\), \(q \neq q'\) and set \(r := \frac{\log q}{\log q'}\). For all \(0 \neq x \in K\), we have
\[|x|_q = q^{-v(x)} = q'^{-rv(x)} = \left|x\right|^{r}_q .\]
Consequently, \(| |_q\) and \(| |_{q'}\) are equivalent.

From now on, when we will deal with fields with a non-Archimedean absolute value, according to theorem 2, we will just speak of a field with a valuation.

**Remark on terminology:** Note that some authors use the term “exponential valuation” rather than “valuation”. In this case the term “valuation” means “absolute value”.

### 1.4 Valuation ring and residue field

**Theorem 3.** Let \(K\) be a field, \(v\) be a valuation on \(K\) and denote by \(| |\) a corresponding non-Archimedean absolute value. Then:

1. the set
\[
\mathcal{O} := \{x \in K \mid v(x) \geq 0\} = \{x \in K \mid |x| \leq 1\}
\]
is an integral domain and a maximal proper subring of \(K\), called the **valuation ring**; moreover, for all \(0 \neq x \in K\), we have that \(x \in \mathcal{O}\) or \(x^{-1} \in \mathcal{O}\),

2. the set
\[
\mathcal{O}^* := \{x \in K \mid v(x) = 0\} = \{x \in K \mid |x| = 1\}
\]
is the group of units of \(\mathcal{O}\),
3. the set
\[ p := \mathcal{O} \setminus \mathcal{O}^* = \{ x \in K \mid v(x) > 0 \} = \{ x \in K \mid |x| < 1 \} = \{ x \in \mathcal{O} \mid x^{-1} \not\in \mathcal{O} \} \]

is the unique maximal ideal of \( \mathcal{O} \).

A ring that has a unique maximal ideal is called a **local ring**. By 3., \( \mathcal{O} \) is a local ring. Besides, two equivalent valuations (or two equivalent non-Archimedean absolute values) on \( K \) give the same valuation ring.

**Proof:** We just consider the valuation \( v \) on \( K \), since by theorem 2 the valuations and the non-Archimedean absolute values are closely related.

It is easy to check that if two valuation \( v \) and \( v' \) on \( K \) are equivalent, i.e., there is a positive real \( s \) such that \( v = sv' \), the sets \( \mathcal{O}, \mathcal{O}^* \) and \( p \) are the same since if \( v(x) \geq 0 \), resp. \( v(x) = 0 \), resp. \( v'(x) = s^{-1}v(x) \geq 0 \), resp. \( v'(x) = s^{-1}v(x) > 0 \).

To prove that \( \mathcal{O} \) is a integral domain, it suffices to show that is closed under addition and multiplication and that every element in \( \mathcal{O} \) has an additive inverse in \( \mathcal{O} \), since all the remaining properties are verified on the field \( K \) and therefore also for the subset \( \mathcal{O} \). Clearly, \( 0, 1 \in \mathcal{O} \) since \( v(0) = \infty > v(1) = 0 \geq 0 \). Take \( 0 \neq x \in \mathcal{O} \) and so \( -x \in \mathcal{K} \). We have \( \infty = v(0) = v(x - x) > \min \{ v(x), v(-x) \} \) since \( x \neq 0 \) (and therefore \( -x \neq 0 \)). By property 3. of Lemma 2, this implies that \( v(-x) = v(x) \geq 0 \), hence \( -x \in \mathcal{O} \). Let \( x, y \in \mathcal{O} \). Then \( v(xy) = v(x) + v(y) \geq 0 \) and so \( xy \in \mathcal{O} \).

Similarly, \( v(x + y) \geq \min \{ v(x), v(y) \} \geq 0 \) and so \( x + y \in \mathcal{O} \). Take now \( 0 \neq x \in \mathcal{K} \).

If \( v(x) \geq 0 \), \( x \in \mathcal{O} \), and if \( v(x) < 0 \), \( v(x^{-1}) = -v(x) > 0 \) and hence \( x^{-1} \in \mathcal{O} \). Let now be \( z \in K \setminus \mathcal{O} \). We want to show that \( \mathcal{O}[z] = K \) in order to prove that \( \mathcal{O} \) is a maximal proper subring of \( K \). Since \( z \notin \mathcal{O}, z^{-1} \in \mathcal{O} \) and \( v(z^{-1}) > 0 \). Take an element \( y \in K \), then there exists \( k \in \mathbb{N} \) such that \( v(yz^{-k}) \geq 0 \) since \( v(z^{-1}) > 0 \).

Consequently, \( w := yz^{-k} \in \mathcal{O} \) and finally \( y = wz^k \in \mathcal{O}[z] \) and so \( \mathcal{O}[z] = K \).

Assume now \( x \in \mathcal{O} \) and \( x^{-1} \in \mathcal{O} \). If \( v(x) > 0 \), \( v(x^{-1}) = -v(x) < 0 \), therefore \( v(x) = 0 \). The other inclusion is obvious. This shows that \( \mathcal{O}^* \) is the group of units of \( \mathcal{O} \).

We show now that \( p \) is the unique maximal ideal of \( \mathcal{O} \). Let \( x \in p \) and \( z \in \mathcal{O} \). Then \( v(xz) = v(z) + v(x) \geq v(x) > 0 \) and hence \( zx \in p \). Let now \( x, y \in p \), then \( v(x - y) \geq \min \{ v(x), v(-y) \} \). If \( y = 0 \), \( x - y \in p \). If \( y \neq 0 \), we have that \( \infty = v(0) = v(y - y) > \min \{ v(y), v(-y) \} \) and, as before, \( v(y) = v(-y) > 0 \).

Therefore \( v(x - y) > 0 \). This means that \( x - y \in p \) and hence \( p \) is an additive subgroup of \( \mathcal{O} \). This shows that \( p \) is an ideal of \( \mathcal{O} \). Assume now there exists an ideal \( A \) of \( \mathcal{O} \) such that \( p \varsubsetneq A \). Then there exists \( x \in \mathcal{O}^* \cap A \). Therefore \( 1 \in A \) and \( A = \mathcal{O} \). This shows that \( p \) is maximal. Assume now that there is a maximal ideal \( B \neq \mathcal{O} \) of \( \mathcal{O} \) such that \( B \neq p \). We must have that \( B \cap \mathcal{O}^* = \{ 0 \} \), because if not, \( 1 \in B \) and so \( B = \mathcal{O} \). This implies that \( B \subsetneq p \) and therefore \( B \) is not maximal.

□
The theorem tells us that for two equivalent valuation we have the same valuation ring, so, from now on, if a valuation is discrete, we can assume without loss of generality that it is normalized. Clearly if a result holds for a normalized valuation, it holds also for a discrete valuation.

**Definition 7.** The field $K := \mathcal{O}/p$ is called the **residue field** of $\mathcal{O}$.

We have an interesting property when a valuation is normalized.

**Lemma 3.** Let $v$ be a normalized valuation on $K$, then for all $0 \neq x \in K$ we can write $x = ut^n$, where $t \in p$ with $v(t) = 1$, $u \in \mathcal{O}^*$ and $n \in \mathbb{Z}$. An element $x \in p$ such that $v(x) = 1$ is called **prime element**.

**Proof:** Since $v(K^*) = \mathbb{Z}$, there exists an element $t \in K$ with $v(t) = 1$. Therefore $t \in p$. Let $0 \neq x \in K$. We have that $v(x) = m$ for some $m \in \mathbb{Z}$. Hence $v(xt^{-m}) = 0$ and so $u := xt^{-m} \in \mathcal{O}^*$ and finally $x = ut^m$.

Recall that an ideal of a ring is principal, if it is generated by one element.

**Theorem 4.** Assume $v$ is a normalized valuation on $K$, then $\mathcal{O}$ is a principal ideal domain, i.e., every ideal of $\mathcal{O}$ is principal. Moreover, all the ideals of $\mathcal{O}$ different from $\{0\}$ are of the form $p^n = t^n\mathcal{O} = \{x \in K \mid v(x) \geq n\}$, $n \geq 0$, where $t \in p$, $v(t) = 1$. Furthermore, we have $p^n/p^{n+1} \cong K$, $n \geq 1$, as $K$-vector spaces.

**Proof:** Let $\{0\} \neq A \subseteq \mathcal{O}$ be an ideal of the valuation ring and $0 \neq x \in A$ such that $n := v(x) \leq v(y)$ for all $y \in A$. By lemma 3, $x = ut^n$ for some prime element $t \in p$ and some $u \in \mathcal{O}^*$. This implies that $t^n\mathcal{O} \subseteq A$. Take now $y \in A$. According to lemma 3, $y = wt^n$, $w \in \mathcal{O}^*$. Since $y \in A$, $m = v(y) \geq n = v(x)$, so we can write $y = (wt^m - n)t^n \in t^n\mathcal{O}$, hence $A \subseteq t^n\mathcal{O}$.

Consider now the $K$-homomorphism

$$\phi : \mathcal{O}/p \to \mathcal{O}/p$$

$$at^n \mapsto a \mod p,$$

with $t \in p$ a prime element and $a \in \mathcal{O}$. Clearly, $\phi$ is surjective and its kernel is $p^{n+1}$. Hence, $\phi$ is a $K$-isomorphism.
This theorem gives us a useful lemma.

**Lemma 4.** Let \( v, v' \) be two discrete valuation on \( K \) such that \( O_v = O_{v'} \), where \( O_v \), resp. \( O_{v'} \), denotes the valuation ring generated by \( v \), resp. by \( v' \). Then \( v \) and \( v' \) are equivalent.

**Proof:** Clearly, \( O_v = O_{v'} \) implies \( O_v^* = O_{v'}^* \) and \( p_v = p_{v'} \). By theorem 4, there is an element \( t \in p_v = p_{v'} \) such that \( p_v = tO_v = tO_{v'} = p_{v'} \). Again by theorem 4, \( t \) must have the minimal valuation among the elements of \( p_v = p_{v'} \). Set \( v(t) = s \) and \( v'(t) = s' \) and this gives \( v(K^*) = s\mathbb{Z} \) and \( v'(K^*) = s'\mathbb{Z} \). Clearly, for all \( x \in O_v^* = O_{v'}^* \), \( 0 = v(x) = v'(x) = 0 \). Consider \( x \in K \setminus O_v^* = K \setminus O_{v'}^* \). Then, by lemma 3, \( x = t^n u \) for some \( n \in \mathbb{Z} \) and some \( u \in O_v^* = O_{v'}^* \). So, \( v(x) = ns \) and \( v'(x) = s'n \) and this gives \( v(x) = \frac{s}{s'} v'(x) \). Since \( x \) was arbitrary, the result holds for all \( x \in K \setminus O_v^* = K \setminus O_{v'}^* \). Therefore \( v \) and \( v' \) are equivalent.

\[ \square \]

### 1.5 Example: the field of rational numbers \( \mathbb{Q} \)

In this paragraph, we deal with the field of rational numbers \( \mathbb{Q} \).

We know that on \( \mathbb{Q} \) we have the restriction of the Archimedean absolute value on \( \mathbb{R} \)

\[
| \cdot | : \mathbb{Q} \longrightarrow \mathbb{R} \quad x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}
\]

Since \( | \cdot | \) is Archimedean, all the constructions that we made above are worthless. We want a non-Archimedean absolute value on \( \mathbb{Q} \). Consider a prime \( p > 1 \). For all \( x \in \mathbb{Q} \) we can write

\[
x = p^a \frac{a}{b}
\]

with \( n \in \mathbb{Z}, a \in \mathbb{Z}, 0 \neq b \in \mathbb{N}, (a, b) = 1, p \nmid a \) and \( p \nmid b \). Define the function \( v_p : \mathbb{Q} \rightarrow \mathbb{R} \cup \{\infty\} \) as follows:

\[
v_p(x) = v_p(p^a \frac{a}{b}) = n,
\]

for all \( 0 \neq x \in \mathbb{Q} \), and \( v_p(0) = \infty \). Clearly \( v_p \) is normalized valuation. According to theorem 3, we have the valuation ring

\[
O_p = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, (a, b) = 1, p \nmid b \right\},
\]

its group of units

\[
O_p^* = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, (a, b) = 1, p \nmid a, p \nmid b \right\},
\]

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and its unique maximal ideal
\[ p_p = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, (a, b) = 1, p \nmid a, p \nmid b \right\}. \]

Clearly \( p \in \mathbb{Q} \) is a prime element and therefore all the non-zero ideals of \( \mathcal{O}_p \) are of the form
\[ p^n \mathcal{O}_p, \quad n \geq 0. \]

Let now \( x = p^n \frac{a}{b} \in \mathcal{O}_p, \quad (a, b) = 1, a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, \quad p \nmid a, p \nmid b \) and \( n \in \mathbb{N} \). If \( n > 0 \), \( x \in p_p \) and therefore \( x \equiv 0 \mod p_p \). Suppose \( n = 0 \), then \( x \in \mathcal{O}_p^\ast \). We now that for every \( m \in \mathbb{Z} \), we can write
\[ m = \sum_{i=0}^{r} m_i p^i \]
with \( r, m_i \in \mathbb{N}, \quad 0 \leq m_i < p, \quad 0 \leq i \leq r \). Hence, we have
\[ a = \sum_{i=0}^{s} a_i p^i \quad \text{and} \quad b = \sum_{j=0}^{t} b_j p^j \]
with \( s, t, a_i, b_j \in \mathbb{N}, \quad 0 \leq a_i, b_j < p, \quad 0 \leq i \leq s, \quad 0 \leq j \leq t \). This gives
\[ x = \frac{a}{b} = \frac{a_0 + a_1 p + \cdots + a_s p^s}{b} = \frac{a_0}{b} + \frac{a_1 p + \cdots + a_s p^s}{b} \]
with
\[ \frac{a_1 p + \cdots + a_s p^s}{b} = \frac{a_1 + \cdots + a_s p^{s-1}}{b} \in p_p. \]
Moreover, we have that
\[ \frac{a_0}{b} = \frac{a_0}{b_0 + b_1 p + \cdots + b_t p^t} \]
and so
\[ \frac{a_0 b_1 p + a_0 b_2 p^2 + \cdots + a_0 b_t p^t}{b_0^2 + b_0 b_1 p + \cdots + b_0 b_t p^t} \in p_p. \]
This means that
\[ x \equiv \frac{a_0}{b_0} \mod p_p \]
and finally we get that
\[ \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \cong \mathbb{K}_p = \mathcal{O}_p/p_p. \]
By theorem 2, we know that each valuation \( v_p \) gives a corresponding family of equivalent non-Archimedean absolute values. We will denote by \( |x|_p \) the absolute value
\[ |x|_p := p^{-v_p(x)}, \]
for all \( 0 \neq x \in \mathbb{Q} \).

We conclude this example with an important result.
Theorem 5. On \( \mathbb{Q} \), each non-trivial absolute value is equivalent to an absolute value \( | \cdot |_p \) or \( | \cdot |_\infty \).

Proof: Let \( || \cdot || \) be a non-Archimedean absolute value on \( \mathbb{Q} \). We have
\[
||n|| = ||1 + \cdots + 1|| \leq \max \{ ||1||, \ldots , ||1|| \} = 1.
\]
There exists a prime \( p > 1 \) such that \( ||p|| < 1 \). If not, \( || \cdot || \) is the trivial valuation.
Consider the set
\[
A := \{ a \in \mathbb{Z} | ||a|| < 1 \}.
\]
Clearly, \( A \) is an ideal of \( \mathbb{Z} \) and \( p\mathbb{Z} \subseteq A \neq \mathbb{Z} \). Since \( p\mathbb{Z} \) is a maximal ideal, we have that \( A = p\mathbb{Z} \). Take \( a \in \mathbb{Z}, a = p^mb \) with \( p \nmid b \). This implies that \( b \notin A \) and therefore \( ||b|| = 1 \).
This gives
\[
||a|| = ||p^mb|| = ||p||^m = |a|^s_p,
\]
with \( s := -\frac{\log ||p||}{\log p} \). Therefore \( || \cdot || \) is equivalent to \( | \cdot |_p \).
Assume now that \( || \cdot || \) is Archimedean. For all integer \( m, n > 1 \), we have
\[
||m||^{\frac{1}{\log n}} = ||n||^{\frac{1}{\log n}}.
\]
Indeed, we can write
\[
m = a_0 + a_1n + \cdots + a_rn^r
\]
with \( r, a_i \in \mathbb{N}, 0 \leq a_i < n, 0 \leq i \leq r \). Clearly \( n^r \leq m \), and so \( r \leq \frac{\log m}{\log n} \). Moreover,
\[
||a_i|| \leq ||1 + \cdots + 1|| \leq a_i ||1|| \leq n
\]
gives that
\[
||m|| \leq \sum_{i=0}^{r} ||a_i|| ||n||^i \leq \sum_{i=0}^{r} ||a_i|| ||n||^r \leq \left( 1 + \frac{\log m}{\log n} \right) n ||n||^{\frac{\log m}{\log n}}.
\]
Replacing \( m \) by \( m^k, k \in \mathbb{N} \), and taking the \( k \)-th root, we get that
\[
||m|| = k^{\frac{1}{k}}||m^k|| \leq k^{\frac{1}{k}}\left( 1 + \frac{k \log m}{\log n} \right) n ||n||^{\frac{\log m}{\log n}} = ||n||^{\frac{\log m}{\log n}} \sqrt{\left( 1 + \frac{k \log m}{\log n} \right)n},
\]
and when \( k \) goes to \( \infty \) we have
\[
||m|| \leq ||n||^{\frac{\log m}{\log n}} \quad \text{and so} \quad ||m||^{\frac{1}{\log n}} \leq ||n||^{\frac{1}{\log n}}.
\]
Exchanging the roles of \( m \) and \( n \) we get the inverse inequality.
Since \( ||n||^{\frac{1}{\log n}} > 0 \), there exists \( s \in \mathbb{R} \) such that \( e^s = ||n||^{\frac{1}{\log n}} \) and so \( ||n|| = e^{s \log n} \).
Hence, for all \( x \in \mathbb{Q}, x > 0 \), we have
\[
||x|| = e^{s \log x} = x^s = |x|^s
\]
Since \( ||x|| = ||-x|| \), we have that \( || \cdot || \) is equivalent to \( | \cdot | \).
\( \square \)
1.6 Example: the rational function field $\mathbb{F}_q(T)$

Let $q = p^n$, $p > 1$ a prime and $n \in \mathbb{N} \setminus \{0\}$. The field

$$
\mathbb{F}_q(T) = \left\{ \frac{f(T)}{g(T)} \mid f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\} \right\}
$$

is called rational function field.

A polynomial is called monic if the leading coefficient is equal to 1. Let $p(T) \in \mathbb{F}_q[T]$ be a monic, irreducible polynomial. For all $f(T) \in \mathbb{F}_q(T)$ we can write

$$
f(T) = p(T)^n \frac{g(T)}{h(T)},
$$

with $n \in \mathbb{Z}$, $g(T) \in \mathbb{F}_q[T]$ such that $p(T) \nmid g(T)$ and $h(T) \in \mathbb{F}_q[T] \setminus \{0\}$ such that $p(T) \nmid h(T)$.

Define the function $v_{p(T)} : \mathbb{F}_q(T) \to \mathbb{Z} \cup \{\infty\}$ as follows:

$$
v_{p(T)}(f(T)) = v_{p(T)} \left( p(T)^n \frac{g(T)}{h(T)} \right) := n,
$$

for all $0 \neq f(T) \in \mathbb{F}_q(T)$ and $v_{p(T)}(0) = \infty$. Obviously, $v_{p(T)}$ is a normalized valuation and we have, according to theorem 3, the valuation ring

$$
\mathcal{O}_{p(T)} = \left\{ \frac{f(T)}{g(T)} \mid f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \nmid g(T) \right\},
$$

its group of units

$$
\mathcal{O}_{p(T)}^* = \left\{ \frac{f(T)}{g(T)} \mid f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \mid f(T), p(T) \nmid g(T) \right\},
$$

and its unique maximal ideal

$$
\mathfrak{p}_{p(T)} = \left\{ \frac{f(T)}{g(T)} \mid f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \mid f(T), p(T) \nmid g(T) \right\}.
$$

Moreover, the residue field of $\mathcal{O}_{p(T)}$, $\mathcal{K}_{p(T)} = \mathcal{O}_{p(T)}/\mathfrak{p}_{p(T)}$ is isomorphic to $\mathbb{F}_q[T]/(p(T))$.

Indeed, consider the ring homomorphism

$$
\varphi : \mathbb{F}_q[T] \longrightarrow \mathcal{O}_{p(T)}/\mathfrak{p}_{p(T)} \quad \text{mod } \mathfrak{p}_{p(T)}.
$$

Clearly, the kernel of $\varphi$ is the ideal $(p(T))$ generated by $p(T)$ in $\mathbb{F}_q[T]$. Take now $h(T) \in \mathcal{O}_{p(T)}$. We can write $h(T) = \frac{r(T)}{s(T)}$ with $r(T), s(T) \in \mathbb{F}_q[T]$, such that $s(T) \neq 0$ and $p(T) \nmid s(T)$. Thus, there exist $a(T), b(T) \in \mathbb{F}_q[T]$ with $a(T)p(T) + b(T)s(T) = 1$ and therefore

$$
h(T) = 1 \cdot h(T) = \frac{a(T)r(T)}{s(T)} p(T) + b(T)r(T),
$$

and

$$
h(T) = \frac{a(T)r(T)}{s(T)} p(T) + b(T)r(T).
$$
and so
\[ h(T) \equiv b(T)r(T) \mod \mathfrak{p}_{p(T)}. \]
Since \( b(T), r(T) \in \mathbb{F}_q[T], \varphi \) is surjective and we have an isomorphism
\[ \mathbb{F}_q[T]/(p(T)) \cong \mathcal{K}_{p(T)} = \mathcal{O}_{p(T)}/\mathfrak{p}_{p(T)}. \]

Let \( f(T) \in \mathbb{F}_q(T) \). We can write
\[ f(T) = \frac{g(T)}{h(T)}, \]
with \( g(T) \in \mathbb{F}_q[T], h(T) \in \mathbb{F}_q[T] \setminus \{0\} \). Consider now the function
\[ v_{\infty} : \mathbb{F}_q(T) \to \mathbb{Z} \cup \{\infty\} \]
defined as follows:
\[ v_{\infty}(f(T)) = v_{\infty}\left(\frac{g(T)}{h(T)}\right) := \deg h(T) - \deg g(T), \]
for all \( 0 \neq f(T) \in \mathbb{F}_q(T) \) and \( v_{\infty}(0) = \infty \). Obviously, \( v_{\infty} \) is a normalized valuation and, by theorem 3, we have the valuation ring
\[ \mathcal{O}_{\infty} = \left\{ \frac{f(T)}{g(T)} \middle| f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg f(T) \leq \deg g(T) \right\}, \]
its group of units
\[ \mathcal{O}^*_{\infty} = \left\{ \frac{f(T)}{g(T)} \middle| f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg f(T) = \deg g(T) \right\}, \]
and its unique maximal ideal
\[ \mathfrak{p}_{\infty} = \left\{ \frac{f(T)}{g(T)} \middle| f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg f(T) < \deg g(T) \right\}. \]
We have that \( T^{-1} \in \mathbb{F}_q(T) \) is a prime element, since \( v_{\infty}(T^{-1}) = \deg T - \deg 1 = 1 \).
Therefore, all non-zero ideal are of the form
\[ \mathfrak{p}_{\infty}^n = (T)^{-n} \mathcal{O}_{\infty}, \ n \geq 0. \]
Take \( f(T) \in \mathcal{O}_{\infty}, \ f(T) = \frac{g(T)}{h(T)}, \) with \( g(T) \in \mathbb{F}_q[T], h(T) \in \mathbb{F}_q[T] \setminus \{0\} \) and \( n := \deg g(T) \leq \deg h(T) =: m. \) Then we can write
\[ g(T) = a_nT^n + a_{n-1}T^{n-1} + \cdots + a_0 \text{ and } h(T) = b_mT^m + b_{m-1}T^{m-1} + \cdots + b_0, \]
with \( a_i \in \mathbb{F}_q, \ 0 \leq i \leq n, \ a_n \neq 0, \) and \( b_j \in \mathbb{F}_q, \ 0 \leq j \leq m, \ b_m \neq 0. \) If \( n < m, \) we have
\[ f(T) \equiv 0 \mod \mathfrak{p}_{\infty}. \] If \( n = m, \) we have
\[ f(T) = \frac{g(T)}{h(T)} = \frac{a_nT^n + a_{n-1}T^{n-1} + \cdots + a_0}{b_mT^m + b_{m-1}T^{m-1} + \cdots + b_0} + \frac{a_{n-1}T^{n-1} + \cdots + a_0}{b_nT^n + b_{n-1}T^{n-1} + \cdots + b_0}. \]
Definition 8. Let \( v \) be a valuation on \( \mathbb{F}_q(T) \) and \( p_v \) the maximal ideal of the valuation ring with respect to \( v \). Then
\[
\deg p_v := [\mathcal{K}_v : \mathbb{F}_q]
\]
is called the degree of \( p_v \).

Note that if a valuation is defined as above by an irreducible, monic polynomial \( p(T) \in \mathbb{F}_q[T] \), the degree of \( p_v(T) \) is \( \deg p_v(T) = \deg p(T) \). Indeed \( \mathcal{K}_{p(T)} \cong \mathbb{F}_q[T]/(p(T)) \) and therefore \( \deg p_v(T) = [\mathcal{K}_{p(T)} : \mathbb{F}_q] = \deg p(T) \). Moreover \( \deg p_\infty = 1 \), since \( \mathcal{K}_\infty \cong \mathbb{F}_q \).

We have an interesting lemma about the degree.

Lemma 5. Let \( v \) be a valuation on \( \mathbb{F}_q(T) \) and \( 0 \neq f(T) \in p_v \). Then we have
\[
\deg p_v \leq [\mathbb{F}_q(T) : \mathbb{F}_q(f(T))] < \infty,
\]
where \( \mathbb{F}_q(f(T)) \) denotes the field generated by \( f(T) \) over \( \mathbb{F}_q \).

Proof: Note that \( \mathbb{F}_q \subset \mathbb{F}_q(f(T)) \). Write \( f(T) = \frac{g(T)}{h(T)} \), with \( g(T), h(T) \in \mathbb{F}_q[T] \).
Then, \( p(X) := f(T)h(X) - g(X) \) is a polynomial in \( X \) with coefficients in \( \mathbb{F}_q(f(T)) \).
Clearly \( T \) is a root of \( p(X) \), hence \( [\mathbb{F}_q(T) : \mathbb{F}_q(f(T))] < \infty \).
For the remaining inequality, it suffices to show that any \( f_1(T), \ldots, f_n(T) \in \mathcal{O}_v \), whose residue class in \( \bar{f}_1(T), \ldots, \bar{f}_n(T) \in \mathcal{K}_v \) are linearly independent over \( \mathbb{F}_q \), are linearly independent over \( \mathbb{F}_q(f(T)) \). Assume that
\[
\sum_{i=0}^{n} \varphi_i f_i(T) = 0
\]
This contradicts the linear independence of $\bar{\varphi}$ and $\varphi$ polynomials in $f(T)$ and not all of the $\varphi_i$ are divisible by $f(T)$, so we can write $\varphi_i = a_i + f(T)g_i$, with $a_i \in \mathbb{F}_q$ not all zero, $g_i \in \mathbb{F}_q[f(T)]$, $0 \leq i \leq n$. Since $f(T) \in p_v$ and $g_i \in \mathbb{F}_q[f(T)] \subseteq \mathcal{O}_v$, we have that $\varphi_i \equiv a_i \mod p_v$, for all $0 \leq i \leq n$. Hence

$$0 \equiv \sum_{i=0}^n \varphi_i f_i(T) \equiv \sum_{i=0}^n a_i \bar{f}_i(T) \mod p_v.$$ 

This contradicts the linear independence of $\bar{f}_1(T), \ldots, \bar{f}_n(T) \in \mathcal{K}_v$ over $\mathbb{F}_q$.

\[ \square \]

We can state now a very important theorem concerning the rational function field $\mathbb{F}_q(T)$.

**Theorem 6.** All the non-trivial valuations on the rational function field $\mathbb{F}_q(T)$ are equivalent to a valuation $v_{p(T)}$, for $p(T) \in \mathbb{F}_q[T]$ a monic, irreducible polynomial, or to $v_\infty$.

**Proof:** By lemma 4, it suffices to show that if $v$ is a non-trivial valuation different from $v_\infty$, there is an irreducible, monic polynomial $p(T) \in \mathbb{F}_q[T]$ such that $\mathcal{O}_{p(T)} = \mathcal{O}_v$.

Assume first that $T \in \mathcal{O}_v$, then, obviously, $\mathbb{F}_q[T] \subseteq \mathcal{O}_v$. Set $I := \mathbb{F}_q[T] \cap p_v$; this is a prime ideal of $\mathbb{F}_q[T]$. Indeed $p_v$ is a maximal ideal, therefore is prime, and $\mathbb{F}_q[T]$ is a ring. The map $\mathbb{F}_q[T] \rightarrow \mathcal{K}_v$ induces an embedding $\mathbb{F}_q[T]/I \hookrightarrow \mathcal{K}_v$, and therefore, by lemma 5, $I \neq \{0\}$. Indeed, if $I = \{0\}$, we have that $\mathbb{F}_q[T]/I = \mathbb{F}_q[T]$ and so $\mathbb{F}_q[T] \hookrightarrow \mathcal{K}_v$. But $\infty = [\mathbb{F}_q[T]:\mathbb{F}_q] \leq [\mathcal{K}_v : \mathbb{F}_q] < \infty$. Since $I \neq \{0\}$ and $I$ is prime, there is a unique irreducible, monic polynomial $p(T) \in \mathbb{F}_q[T]$ such that $I = p(T)\mathbb{F}_q[T]$ (recall that the ring $\mathbb{F}_q[T]$ is principal). Any $g(T) \in \mathbb{F}_q[T]$ with $p(T) \nmid g(T)$ is not in $I$, so $g(T) \notin p_v \subset \mathcal{O}_v$, therefore $g(T)^{-1} \in \mathcal{O}_v$. Hence, we have

$$\mathcal{O}_{p(T)} = \left\{ \frac{f(T)}{g(T)} \mid f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \nmid g(T) \right\} \subseteq \mathcal{O}_v.$$ 

By theorem 3, all valuation rings are maximal proper subrings of $\mathbb{F}_q(T)$, therefore $\mathcal{O}_{p(T)} = \mathcal{O}_v$. Therefore $v$ and $v_{p(T)}$ are equivalent.

Assume now that $T \notin \mathcal{O}_v$. We have that $\mathbb{F}_q[T^{-1}] \subseteq \mathcal{O}_v$, $T^{-1} \in p_v \cap \mathbb{F}_q[T^{-1}]$ and clearly $p_v \cap \mathbb{F}_q[T^{-1}] = T^{-1}\mathbb{F}_q[T^{-1}]$, since $p_v \cap \mathbb{F}_q[T^{-1}]$ is a prime ideal. As before, if $g(T^{-1}) \in \mathbb{F}_q[T^{-1}]$ with $T^{-1} \nmid g(T^{-1})$, $g(T^{-1}) \notin p_v$ and so $g(T^{-1})^{-1} \in \mathcal{O}_v$. This
gives
\[ O_v \supseteq \left\{ \frac{f(T^{-1})}{g(T^{-1})} \left| f(T^{-1}) \in \mathbb{F}_q[T^{-1}], g(T^{-1}) \in \mathbb{F}_q[T^{-1}] \setminus \{0\}, T^{-1} \nmid g(T^{-1}) \right. \right\} \]
\[ = \left\{ \frac{a_0 + a_1 T^{-1} + \cdots + a_n T^{-n}}{b_0 + b_1 T^{-1} + \cdots + b_m T^{-m}} \left| b_0 \neq 0 \right. \right\} \]
\[ = \left\{ \frac{a_0 T^{m+n} + a_1 T^{m+n-1} + \cdots + a_n T^m}{b_0 T^{m+n} + b_1 T^{m+n-1} + \cdots + b_m T^n} \left| b_0 \neq 0 \right. \right\} \]
\[ = \left\{ \frac{u(T)}{v(T)} \left| u(T) \in \mathbb{F}_q[T], v(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg u(T) \leq \deg v(T) \right. \right\} \]
\[ = O_\infty. \]

According to theorem 3, this implies that \( O_v = O_\infty \) and so \( v \) is equivalent to \( v_\infty \).

\[ \square \]

This theorem and the theorem 5 for \( \mathbb{Q} \) are very similar. The valuations \( v_p \), given by a prime \( p \), on \( \mathbb{Q} \) correspond to the valuations \( v_{p(T)} \), given by a monic, irreducible polynomial \( p(T) \), on \( \mathbb{F}_q(T) \) and we have a difference between the usual absolute value \( | \cdot | \) on \( \mathbb{Q} \) and the valuation \( v_\infty \) on \( \mathbb{F}_q(T) \). The problem is that \( | \cdot | \) is Archimedean, hence we haven’t any corresponding valuation.

We conclude this example with a corollary.

**Corollary.** There is a bijection between the equivalence classes of non-trivial valuations \( v \) on \( \mathbb{F}_q(T) \) with \( \deg p_v = 1 \) and \( \mathbb{F}_q \cup \{\infty\} \).

**Proof:** By theorem 6, the valuations \( v_{p(T)}, p(T) \) an irreducible, monic polynomial in \( \mathbb{F}_q[T] \), and \( v_\infty \) represent all the equivalence classes of non-trivial valuations on \( \mathbb{F}_q(T) \). Moreover, by the note after the definition 8, the valuations with degree 1 are exactly \( v_\infty \) and \( v_{T-\alpha} \), for all \( \alpha \in \mathbb{F}_q \). Therefore, we have a bijection between \( \mathbb{F}_q \cup \{\infty\} \) and the equivalence classes of non-trivial valuations on \( \mathbb{F}_q(T) \).

\[ \square \]

**Remark:** In fact all the results that holds for \( \mathbb{F}_q(T) \), holds also for all fields

\[ K(T) := \left\{ \frac{f(T)}{g(T)} \left| f(T) \in K[T], g(T) \in K[T] \setminus \{0\} \right. \right\}, \]

where \( K[T] \) denotes the ring of polynomials over a field \( K \). \( K(T) \) is called rational function field.
2 Completion

2.1 Definitions and results

We begin this section with some basic definitions.

**Definition 9.** Let $K$ be a field and $| |$ an absolute value on $K$. A sequence $\{a_n\}_{n \in \mathbb{N}}$ in $K$ is called a **Cauchy sequence** if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \implies |a_n - a_m| < \varepsilon.$$ 

Another important

**Definition 10.** A field $K$ with an absolute value $| |$ is called **complete** if any Cauchy sequence $\{a_n\}_{n \in \mathbb{N}}$ in $K$ converges to an element $a \in K$, i.e.

$$\lim_{n \to \infty} |a_n - a| = 0.$$ 

A useful lemma for non-Archimedean absolute values.

**Lemma 6.** Let $K$ be a complete field and $| |$ a non-Archimedean absolute value on $K$. Then, for $\{a_n\}_{n \in \mathbb{N}} \subset K$, we have:

1. the sequence $\{a_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if, and only if, $\lim_{n \to \infty} (a_{n+1} - a_n) = 0$,

2. the series $\sum_{n=0}^{\infty} a_n$ converges if, and only if, $\lim_{n \to \infty} a_n = 0$,

3. Suppose that $\lim_{n \to \infty} a_n = a \neq 0$, then there exists a positive integer $N$ such that for all $m \geq N$, $|a_m| = |a_N| = |a|$.

**Proof:**

1. Assume that $\{a_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_m - a_n| < \varepsilon$, therefore we have also, for all $n \geq N$, $|a_{n+1} - a_n| < \varepsilon$. Conversely, assume that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \geq N$, $|a_{n+1} - a_n| < \varepsilon$. Then, for all $r, s \geq N$,

$$|a_r - a_s| = |\sum_{i=n}^{m-1} (a_{i+1} - a_i)|$$

$$\leq \max_{n \leq i < m} \{|a_{i+1} - a_i|\} < \varepsilon.$$ 

Therefore, $\{a_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

2. We have that $|a_n| = \left|\sum_{i=0}^{n} a_i - \sum_{j=0}^{n-1} a_j\right|$. Using 1., the equivalence is obvious.
3. Since \( \lim_{n \to \infty} a_n = a \neq 0 \), there exists a positive integer \( n_a \) such that, for all \( n \geq n_a \), \( |a_n - a| < |a| \). Then, using the note we made after definition 3, we have that \( |a_n| = |a_n - a + a| = \max \{|a_n - a|, |a|\} = |a| \).

\[
\exists a > 0 \text{ such that } |x| < a \leftrightarrow \exists b \in \mathbb{R} \text{ such that } v(x) > b,
\]

with \( v \) a corresponding valuation. Indeed, for \( s > 0 \), if \( |x| < a \), \( v(x) = -s \log |x| > -s \log a \). Conversely, for \( q > 1 \), if \( v(x) > b \), \( |x| = q^{-v(x)} < q^{-b} \).

**Theorem 7.** Let \( K \) be a field and \( | \cdot | \) be an absolute value on \( K \). Then, there exists a unique, up to \( K \)-isomorphism, complete field \( \hat{K} \) with an absolute value \( | \cdot |_{\hat{K}} \) such that \( K \) is embedded in \( \hat{K} \) as a dense subfield and the absolute value on \( K \) is a restriction of the absolute value on \( \hat{K} \), i.e., \( |x|_{\hat{K}} = |x| \) if \( x \in K \).

**Sketch of the proof:** We will not prove this theorem in detail, not because it is too difficult, but because we would need to prove a lot of uninteresting little claims, that can be easily proved by the reader.

Let \( R \) be the set of all the Cauchy sequences in \( K \) with respect to \( | \cdot | \). Define the addition and the multiplication as follows, for all \( \{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}} \in R \):

\[
\{a_n\}_{n \in \mathbb{N}} + \{b_n\}_{n \in \mathbb{N}} := \{a_n + b_n\}_{n \in \mathbb{N}} \quad \text{and} \quad \{a_n\}_{n \in \mathbb{N}} \cdot \{b_n\}_{n \in \mathbb{N}} := \{a_n b_n\}_{n \in \mathbb{N}}.
\]

Indeed \( R \) is a ring. Let \( m \subset R \) be the set of all the Cauchy sequences that converge to 0. It not difficult to prove that \( m \) is a maximal ideal of \( R \). Now set

\[
\hat{K} := R/m.
\]

Clearly, \( \hat{K} \) is a field. We have an injection \( K \hookrightarrow \hat{K} \) by sending \( a \in K \) to the equivalence class of the Cauchy sequence \( (a, a, a, \ldots) \). Hence, we can write \( K \subset \hat{K} \).

Take \( a \in \hat{K} \) and let \( \{a_n\}_{n \in \mathbb{N}} \in R \) be a representative of \( a \). Then, we have that the sequence \( \{a_n\}_{n \in \mathbb{N}} \) converges in \( \mathbb{R} \), because it is a Cauchy sequence, since \( ||a_n| - |a_m|| \leq |a_n - a_m| \), by 4. of lemma 1. Set

\[
|a|_{\hat{K}} := \lim_{n \to \infty} |a_n|,
\]

then \( | \cdot |_{\hat{K}} \) is an absolute value on \( \hat{K} \) and, if \( a \in K \), we have \( |a|_{\hat{K}} = |a| \). Furthermore,

\[
\lim_{n \to \infty} a_n = a
\]

in \( \hat{K} \), therefore \( K \) is dense in \( \hat{K} \) and \( \hat{K} \) is complete with respect to \( | \cdot |_{\hat{K}} \).

Let \( \hat{K}' \) be another complete field, with respect to an absolute value \( | \cdot |_{\hat{K}'} \), such that \( K \) is dense in \( \hat{K}' \) and, for all \( x \in K \), \( |x|_{\hat{K}'} = |x| \). Take \( a \in \hat{K} \) and let \( \{a_n\}_{n \in \mathbb{N}} \subset K \) be a representative of \( a \). Then, in \( \hat{K}' \), this Cauchy sequence converges to an element \( a' \in \hat{K}' \), because \( K \) is dense in \( \hat{K}' \). Define the function \( \sigma : \hat{K} \to \hat{K}' \) by \( \sigma(a) := a' \).

It is easy to verify that \( \sigma \) is a \( K \)-isomorphism. Furthermore, \( |a|_{\hat{K}} = |\sigma(a)|_{\hat{K}'} \), because

\[
|a|_{\hat{K}} = \lim_{n \to \infty} |a_n| = \lim_{n \to \infty} |a_n|_{\hat{K}'} = |a|_{\hat{K}'}.
\]
Definition 11. The field $\hat{K}$ is called the **completion** of $K$.

Now we look at an Archimedean absolute value on a field $K$. The following theorem is due to Alexander Ostrowski (1893-1986). We will not prove this theorem.

**Theorem 8.** Let $K$ be a complete field with respect to an Archimedean absolute value $| \cdot |_K$, then there is an isomorphism $\sigma$ from $K$ to $\mathbb{R}$ or $\mathbb{C}$ such that $|x|_K = |\sigma(x)|^s$ for all $x \in K$ and a fixed $0 \leq s \leq 1$, where $| \cdot |$ denotes the usual absolute value of $\mathbb{R}$ or $\mathbb{C}$.

Ostrowski’s theorem tells us that all complete fields with respect to an Archimedean absolute value are isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Therefore, the completion of a field with an Archimedean absolute value is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

Let us look now at the completion of a field with a non-Archimedean absolute value. As seen before, there is a valuation $v$ corresponding to the non-Archimedean absolute value on $K$. In this case, we denote $\hat{v}$ the valuation of the completion $\hat{K}$ of $K$.

Clearly, if $v$ is discrete, resp. normalized, $\hat{v}$ is also discrete, resp. normalized.

**Theorem 9.** Let $K$ be a field, $\hat{K}$ its completion with respect to the valuation $v$ on $K$. Denote $\hat{v}$ the corresponding valuation on $\hat{K}$, $\hat{O}$ resp. $\hat{\mathcal{O}}$ the valuation ring of $K$, resp. $\hat{K}$, $\mathfrak{p}$, resp. $\hat{\mathfrak{p}}$, the maximal ideal of $O$, resp. $\hat{O}$ and $K$, resp. $\hat{K}$, the residue field of $O$, resp. $\hat{O}$. Then

$$\mathcal{O} \cong \hat{K}$$

and, if $v$ is discrete,

$$\mathcal{O}/\mathfrak{p}^n \cong \hat{O}/\hat{\mathfrak{p}}^n, \ n \geq 1.$$

**Proof:** By theorem 7, we have $K \subset \hat{K}$, $\mathcal{O} \subset \hat{O}$ and $\mathfrak{p} \subset \hat{\mathfrak{p}}$. The inclusion $\mathcal{O} \subset \hat{O}$ gives a homomorphism

$$\varphi : \mathcal{O} \rightarrow \hat{O}/\hat{\mathfrak{p}},$$

whose kernel is obviously $\mathfrak{p}$. Let now $x \in \hat{O}$. By theorem 7, $K$ is dense in $\hat{K}$, therefore there is a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \hat{K}$ which converges to $x \in \hat{K}$. Since $\hat{v}(x) \geq 0$, by lemma 6, there exists a positive integer $N$ such that $\hat{v}(x_n) = \hat{v}(x)$, for all $n \geq N$. Hence, we can assume that $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{O}$. By definition, for all $\xi \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that, for all $n \geq N$, $\hat{v}(x - x_n) > \xi$. Take $\xi > 0$, then $x - x_n \in \hat{\mathfrak{p}}$, and we get $x \equiv x_n \mod \hat{\mathfrak{p}}$. This means that $\varphi$ is surjective and therefore we have an isomorphism

$$\mathcal{O}/\mathfrak{p} \cong \hat{O}/\hat{\mathfrak{p}}.$$

Moreover, if $v$ is discrete, $\hat{v}$ is discrete and all the ideal of $\mathcal{O}$, resp. $\hat{\mathcal{O}}$, are of the form $\mathfrak{p}^n$, resp. $\hat{\mathfrak{p}}^n$, $n \geq 1$. So we have a homomorphism $\lambda : \mathcal{O} \rightarrow \hat{O}/\hat{\mathfrak{p}}^n$, whose kernel is $\mathfrak{p}^n$. By the same argument as above, for all $x \in \hat{O}$, there is an element $y_n \in \mathcal{O}$ such that $\hat{v}(x - y_n) \geq n$, for all $n \geq 1$. Therefore $x \equiv y_n \mod \hat{\mathfrak{p}}^n$. Hence, $\lambda$ is surjective and we have an isomorphism $\mathcal{O}/\mathfrak{p}^n \cong \hat{O}/\hat{\mathfrak{p}}^n$.\[\square\]
**Theorem 10.** Take the same assumption as in the preceding theorem and assume that \( v \) is normalized. Let \( R \subseteq O \) be a set of representatives of \( \mathcal{K} \) such that \( 0 \in R \) and let \( t \in p \) be a prime element. Then we can represent all \( x \in \hat{K}^* \) as a converging series

\[
x = t^m(a_0 + a_1t + a_2t^2 + \ldots)
\]

with \( a_i \in R, \ i \in \mathbb{N}, \ a_0 \neq 0 \) and \( m \in \mathbb{Z} \).

**Proof:** Since \( p \subset \hat{p}, \ t \in \hat{p} \) and \( 1 = v(t) = \hat{v}(t) \), according to theorem 7. From now on, in this proof, we will use an absolute value corresponding to the valuation \( \hat{v} \). By lemma 3, we have that \( x = ut^m, \ u \in \hat{O}^* \). Since \( O/p \cong \hat{O}/\hat{p}, \ u \mod \hat{p} \) has a representative \( 0 \neq a_0 \in R \) and therefore we can write \( u = a_0 + tb_1 \) with \( b_1 \in \hat{O} \). By the same argument, we find also \( a_1, a_2, \ldots, a_{n-1} \in R \) such that

\[
u = a_0 + a_1t + \cdots + a_{n-1}t^{n-1} + t^n b_n
\]

with \( b_n \in \hat{O} \). As before, there is an \( a_n \in R \) such that \( b_n = a_n + t b_{n+1}, \ b_{n+1} \in \hat{O} \). Hence,

\[
u = a_0 + a_1t + \cdots + a_{n-1}t^{n-1} + a_n t^n + t^{n+1} b_{n+1}
\]

We can do this for all \( n \in \mathbb{N} \), therefore we have a series

\[
\sum_{r=0}^{\infty} a_r t^r.
\]

It remains to show that this series converges to \( \nu \). For all \( n \in \mathbb{N} \), we have

\[
\hat{v}(\nu - \sum_{i=0}^{n} a_i t^i) = \hat{v}(t^{n+1} b_{n+1}) = \hat{v}(t^{n+1}) + \hat{v}(b_{n+1}) = n + 1 + \hat{v}(b_{n+1}) \geq n + 1,
\]

since \( b_{n+1} \in \hat{O} \). This gives

\[
\lim_{n \to \infty} \hat{v}(\nu - \sum_{i=0}^{n} a_i t^i) = \infty
\]

and hence the series converges to \( \nu \). Finally, we can write

\[
x = ut^m = t^m(a_0 + a_1t + a_2t^2 + \ldots)
\]

\( \square \)

We will state some results, without proof, concerning polynomials. Let \( K \) be a complete field with respect to the valuation \( v \). We can extend \( v \) to the ring \( K[x] \) of the polynomials in one variable over \( K \) as follows:

\[
v(f) := \min \{v(a_0), \ldots, v(a_n)\},
\]

where \( f(x) = a_0 + a_1 x + \cdots + a_n x^n, \ a_i \in K, \ 0 \leq i \leq n, \ a_n \neq 0 \). A polynomial \( f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathcal{O}[x] \) is called **primitive** if \( v(f) = 0 \), i.e., \( f(x) \equiv 0 \mod p \). The following lemma is due to Kurt Hensel (1861-1941).
Lemma 7. Let \( f(x) \in \mathcal{O}[x] \) be a primitive polynomial. Assume that

\[
f(x) \equiv \tilde{g}(x) \tilde{h}(x) \pmod{p},
\]

with \( \tilde{g}(x), \tilde{h}(x) \in K[x] \). Then, there exists two polynomials \( g(x), h(x) \in \mathcal{O}[x] \) with \( \deg g(x) = \deg \tilde{g}(x) \) and

\[
g(x) \equiv \tilde{g}(x) \pmod{p} \quad \text{and} \quad h(x) \equiv \tilde{h}(x) \pmod{p},
\]

such that

\[
f(x) = g(x)h(x).
\]

We have an immediate consequence.

Corollary. For all irreducible polynomial \( f(x) = a_0 + a_1x + \cdots + a_nx^n \in K[x] \), we have

\[
v(f) = \min \{v(a_0), v(a_n)\}.
\]

Moreover, if \( a_n = 1 \) and \( a_0 \in \mathcal{O}, f \in \mathcal{O}[x] \).

2.2 Example: the field of \( p \)-adic numbers \( \mathbb{Q}_p \)

In this example we deal, as in paragraph 1.5, with the field of rational numbers \( \mathbb{Q} \). By theorem 5, we know that the equivalence classes of absolute values on \( \mathbb{Q} \) are represented by \( | \cdot |_p \), \( p > 1 \) prime, and \( | \cdot | \). Theorem 8 tells us that the completion of \( \mathbb{Q} \) with respect to \( | \cdot |_p \) is isomorphic to \( \mathbb{R} \) or \( \mathbb{C} \), since \( | \cdot | \) is Archimedean. In fact, we know that this completion is \( \mathbb{R} \) (one way to define \( \mathbb{R} \) is to complete \( \mathbb{Q} \) with respect to the usual absolute value).

We are more interested in non-Archimedean absolute values. Hence, let \( p > 1 \) be a prime number. The completion of \( \mathbb{Q} \) with respect to \( | \cdot |_p \) is denoted \( \mathbb{Q}_p \), and called the field of \( p \)-adic numbers. Instead of \( | \cdot |_p \), we use the corresponding valuation \( v_p \); we use the notation \( v_p \) also for the extension of \( v_p \) in \( \mathbb{Q}_p \). We know that \( \mathcal{K}_p \cong \mathbb{Z}/p\mathbb{Z} \), therefore we can take \( \{0, \ldots, p-1\} \) as set of representatives of \( \mathcal{K}_p \); furthermore, \( p \) is a prime element. According to theorem 10, for all \( 0 \neq x \in \mathbb{Q}_p \), we have

\[
x = p^m(a_0 + a_1p + a_2p^2 + \cdots) = \sum_{i=m}^{\infty} a_ip^i,
\]

with \( a_i \in \{0, \ldots, p-1\}, i \in \mathbb{N}, a_0 \neq 0 \) and \( m \in \mathbb{Z} \). By the construction we made in the proof of theorem 10, we know that

\[
u := a_0 + a_1p + a_2p^2 + \cdots = \sum_{i=0}^{\infty} a_ip^i\]

is a unit, i.e., \( v_p(u) = 0 \). This means that \( v_p(x) = m \). Therefore, the valuation ring of \( \mathbb{Q}_p \) is

\[
\mathbb{Z}_p := \left\{ \sum_{i=m}^{\infty} a_ip^i \left| a_i \in \{0, \ldots, p-1\}, a_0 \neq 0, \ m \geq 0 \right. \right\},
\]

21
called the **ring of $p$-adic integers**. Its group of units is

$$\mathbb{Z}_p^* = \left\{ \sum_{i=m}^{\infty} a_i \in \{0, \ldots, p-1\}, a_0 \neq 0, m = 0 \right\}$$

and the unique maximal ideal is $p\mathbb{Z}_p$. Moreover, the residue field of $\mathbb{Z}_p$ is $\mathbb{Z}/p\mathbb{Z}$ since it is isomorphic to the residue field of $\mathcal{O}_p$.

**Remark on notation:** Note that sometimes in topology the notation $\mathbb{Z}_p$ stands for the finite field $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

### 2.3 Example: the field of Laurent series $\mathbb{F}_q((T^{-1}))$

As before, we continue the example of section 1, paragraph 1.6. Let $q = p^n$, $p > 1$ prime, $n \in \mathbb{N} \setminus \{0\}$ and let $\mathbb{F}_q(T)$ be the rational function field. By theorem 6, we know that all equivalence classes of valuation on $\mathbb{F}_q(T)$ are represented by $v_p(T)$, $p(T) \in \mathbb{F}_q[T]$ a monic, irreducible polynomial, and $v_\infty$.

Now, we are going to see what happens when we complete $\mathbb{F}_q(T)$ with respect to the absolute value corresponding to $v_\infty$. We know that $T^{-1}$ is a prime element for $v_\infty$ and that the residue field $K_\infty$ of the valuation ring $\mathcal{O}_\infty$ is isomorphic to $\mathbb{F}_q$, then, by theorem 10, we can write all element $f \neq 0$ of the completion in the form

$$f = (T^{-1})^m \left(a_0 + a_1 T^{-1} + a_2 (T^{-1})^2 + \ldots \right) = (T^{-1})^m \sum_{i=0}^{\infty} a_i (T^{-1})^i$$

with $a_i \in \mathbb{F}_q$, $i \in \mathbb{N}$, $a_0 \neq 0$ and $m \in \mathbb{Z}$. In fact, we abuse of notations and we write

$$f = f(T) = \sum_{i=-\infty}^{-m} a_i T^i, \quad a_{-m} \neq 0.$$

We note the completion of $\mathbb{F}_q(T)$ with respect to $v_\infty$ by $\mathbb{F}_q((T^{-1}))$. Note the analogy between $f(T) \in \mathbb{F}_q((T^{-1}))$ and a Laurent series in $\mathbb{C}$. Recall that a Laurent series in $\mathbb{C}$ is a series that allows infinite negative terms and converges in an annulus. We call $\mathbb{F}_q((T^{-1}))$ the **field of formal Laurent series in $T^{-1}$ over $\mathbb{F}_q$**. As above for $\mathbb{Q}_p$, we use the same notation for the valuation in the completion as in the rational function field. Clearly, we have $v_\infty(f(T)) = m$. The valuation ring of the field of formal Laurent series is the ring

$$\left\{ \sum_{i=-\infty}^{-m} a_i T^i \right| a_i \in \mathbb{F}_q, a_{-m} \neq 0, m \geq 0 \right\}$$

and the units are element of the form

$$\sum_{i=-\infty}^{0} a_i T^i,$$

with $a_0 \neq 0$. The unique maximal ideal of the valuation ring is the set of all the series with only negative powers of $T$. Note that the valuation ring coincides with the ring of formal power series in $T^{-1}$, since no negative powers of $T^{-1}$ occur.
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