Absolute values, valuations and completion

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Introduction

During this talk I'll introduce the basic definitions and some results about valuations, absolute values of fields and completions. These notions will give two basic examples: the *rational function field* $\mathbb{F}_q(T)$ and the field $\mathbb{Q}_p$ of the *p-adic numbers*.

1 Absolute values and valuations

All along this section, K denote a field.

1.1 Absolute values

We begin with a well-known

Definition 1. *An **absolute value** of K is a function*

$$| \ | : K \to \mathbb{R}$$

satisfying these properties, $\forall\, x, y \in K$:

1. $|x| = 0 \ \Leftrightarrow \ x = 0$,
2. $|x| \geq 0$,
3. $|xy| = |x|\,|y|$,
4. $|x + y| \leq |x| + |y|$. (Triangle inequality)

Note that if we set $|x| = 1$ for all $0 \neq x \in K$ and $|0| = 0$, we have an absolute value on K, called the *trivial absolute value.* From now on, when we speak about an absolute value $| \ |$, we assume that $| \ |$ is non-trivial. Moreover, if we define $d : K \times K \to \mathbb{R}$ by $d(x, y) = |x - y|$, $x, y \in K$, d is a metric on K and we have a topological structure on K. We have also some basic properties that we can deduce directly from the definition of an absolute value.

Lemma 1. *Let $| \ |$ be an absolute value on K. We have*

1. $|1| = 1$,
2. $|\zeta| = 1$, *for all* $\zeta \in K$ *with* $\zeta^d = 1$ *for some* $0 \neq d \in \mathbb{N}$ *(ζ is a root of unity),*
3. $|x^{-1}| = |x|^{-1}$,
4. $||x| - |y|| \leq |x - y|$.

Proof:

1. $|1|^2 = |1^2| = |1| \Rightarrow |1| = 1$.
2. $|\zeta|^d = \left|\zeta^d\right| = |1| = 1 \Rightarrow |\zeta| = 1$.
3. $1 = |xx^{-1}| = |x|\,|x^{-1}| \Rightarrow |x^{-1}| = |x|^{-1}$,
4. Set $a := x - y$, $b := y$. We have $|a + b| \leq |a| + |b|$, therefore
   $|x| - |y| = |a + b| - |b| \leq |a| = |x - y|$.

□

Another definition about absolute values.

Definition 2. *Two absolute values on K are* ***equivalent*** *if they define the same topology on K.*

The next theorem and its corollary give us other conditions to verify if two absolute values are equivalent.

Theorem 1. *Let $| \ |_1$ and $| \ |_2$ be two absolute values on K. They are equivalent if, and only if, there exists $s \geq 0$ real such that*

$$|x|_1 = |x|_2^s, \ \forall x \in K.$$

Proof: If we have $|x|_1 = |x|_2^s$, $s > 0$, clearly the two absolute values are equivalent since they define the same open sets.
If $| \ |_1$ and $| \ |_2$ are equivalent, a series converging to 0 with respect to $| \ |_1$ will converge to 0 with respect to $| \ |_2$. Moreover, for all K field and $| \ |$ on K, the inequality $|x| < 1$ is equivalent to saying that the sequence $\{x^n\}_{n \in \mathbb{N}}$ converges to 0. Therefore if $| \ |_1$ and $| \ |_2$ are equivalent, we have

$$|x|_1 < 1 \Leftrightarrow |x|_2 < 1.$$

Let $y \in K$ be an element such that $|y|_1 > 1$ and let $x \in K$, $x \neq 0$. Then there exists $\alpha \in \mathbb{R}$ such that $|x|_1 = |y|_1^\alpha$. Let $\left\{\frac{m_i}{n_i}\right\}_{i\in\mathbb{N}}$ be a sequence of rational numbers ($m_i \in \mathbb{Z}$, $n_i \in \mathbb{N}^*$) converging to α from above. Then $|x|_1 = |y|_1^\alpha < |y|_1^{\frac{m_i}{n_i}}$ and hence

$$\left|\frac{x^{n_i}}{y^{m_i}}\right|_1 < 1 \Rightarrow \left|\frac{x^{n_i}}{y^{m_i}}\right|_2 < 1.$$

This gives $|x|_2 < |y|_2^{\frac{m_i}{n_i}}$ and consequently $|x|_2 \leq |y|_2^\alpha$. Taking a sequence $\left\{\frac{m_i}{n_i}\right\}_{i\in\mathbb{N}}$ converging to α from below will give $|x|_2 \geq |y|_2^\alpha$ and therefore $|x|_2 = |y|_2^\alpha$. So, for all $0 \neq x \in K$, we have

$$\frac{\log |x|_1}{\log |x|_2} = \frac{\log |y|_1}{\log |y|_2} =: s$$

and hence $|x|_1 = |x|_2^s$. Finally, $|y|_1 > 1$ implies $|y|_2 > 1$ and so $s > 0$.

□

Corollary. *Two absolute values $|\ \ |_1$ and $|\ \ |_2$ on K are equivalent if, and only if,*

$$|x|_1 < 1 \iff |x|_2 < 1.$$

We continue with this important

Definition 3. *An absolute value is called **non-Archimedean** if we have*

$$|x + y| \leq \max\{|x|, |y|\}, \ \forall x, y \in K.$$

*Otherwise the absolute value is called **Archimedean**.*

Note that if $|\ \ |$ is non-Archimedean, for $x, y \in K$ we have

$$|x| \neq |y| \Longrightarrow |x + y| = \max\{|x|, |y|\}.$$

Indeed: w.l.o.g we can assume $|x| > |y|$, then obviously $|x + y| \leq \max\{|x|, |y|\} = |x|$. On the other side, $|x| = |x - y + y| \leq \max\{|x + y|, |y|\}$. Assume that $\max\{|x + y|, |y|\} = |y|$, then $|x| \leq |y| < |x|$. This contradicts the hypothesis $|x| > |y|$, hence $\max\{|x + y|, |y|\} = |x + y|$ and so $|x| \leq |x + y|$.

1.2 Valuations

We introduce the symbol ∞ with the convention that for all $a \in \mathbb{R}$ we have $a < \infty$, $a + \infty = \infty$ and $\infty + \infty = \infty$. As for absolute values, we start with a basic

Definition 4. *A **valuation** on K is a function*

$$v : K \to \mathbb{R} \cup \{\infty\}$$

satisfying these properties, for all $x, y \in K$:

1. $v(x) = \infty \iff x = 0$,
2. $v(xy) = v(x) + v(y)$,
3. $v(x + y) \geq \min\{v(x), v(y)\}$.

Note that if we set $v(x) = 0$ for all $0 \neq x \in K$ and $v(0) = \infty$, we have a valuation on K, called the *trivial valuation*. From now on, when we speak about a valuation v, we assume that v is non-trivial.

Now, a lemma with some basic properties induced by the definition of valuation

Lemma 2. *Let v be a valuation on K. We have*

1. $v(1) = 0$,
2. $v(\zeta) = 0$, *for all $\zeta \in K$ root of unity,*
3. $v(x^{-1}) = -v(x), \forall\, x \in K^*$,
4. *if $x, y \in K$ and $v(x) \neq v(y)$, $v(x + y) = \min\{v(x), v(y)\}$.*

Proof:

1. $v(1) = v(1^2) = v(1) + v(1) \Rightarrow v(1) = 0$.
2. Let $\zeta \in K$ with $\zeta^d = 1$ for some $0 \neq d \in \mathbb{N}$. We have $dv(\zeta) = v(\zeta^d) = v(1) = 0$, therefore $v(\zeta) = 0$.
3. $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1}) \Rightarrow v(x^{-1}) = -v(x)$.
4. W.l.o.g. we can assume that $v(x) > v(y)$ and so $y \neq 0$. If $x = 0$, obvious. Assume $x \neq 0$. Then $v(x + y) \geq \min\{v(x), v(y)\} = v(y)$. We have also $v(y) = v(y + x - x) \geq \min\{v(x + y), v(x)\}$. Assume that $\min\{v(x + y), v(x)\} = v(x)$, then $v(x) > v(y) \geq v(x)$ and we get a contradiction. So, $\min\{v(x + y), v(x)\} = v(x + y)$ and finally $v(y) \geq v(x + y)$.

□

Some further terminology regarding valuations.

Definition 5. *A valuation v on K is called **discrete** if $v(K^*) = s\mathbb{Z}$, for a real $s > 0$. Moreover, v is **normalized** if $s = 1$.*

We introduce now the equivalence between valuations.

Definition 6. *Two valuations v_1 and v_2 on K are **equivalent** if there exists a real $s > 0$ such that $v_1 = sv_2$.*

Note that if we have a discrete valuation on K with $v(K^*) = s\mathbb{Z}$, dividing it by s we obtain an equivalent normalized valuation.

1.3 Relations between non-Archimedean absolute values and valuations

The following theorem provides a relation between the non-Archimedean absolute values and the valuations on K.

Theorem 2. *Let $|\ \ |$ be a non-Archimedean absolute value on K and $s \in \mathbb{R}$, $s > 0$, then the function*

$$\begin{array}{rcll} v_s : & K & \longrightarrow & \mathbb{R} \cup \{\infty\} \\ & x & \mapsto & \begin{cases} -s \log |x| & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \end{array}$$

is a valuation on K. Furthermore, if $s, s' \in \mathbb{R}$, $s, s' > 0$ and $s \neq s'$, v_s is equivalent to $v_{s'}$. Conversely, if v is a valuation on K and $q \in \mathbb{R}$, $q > 1$, the function

$$\begin{array}{rcll} |\ \ |_q : & K & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \begin{cases} q^{-v(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{array}$$

is an absolute value on K. Besides, if $q, q' \in \mathbb{R}$, $q, q' > 1$ and $q \neq q'$, $|\ \ |_q$ is equivalent to $|\ \ |_{q'}$.

Proof: We just need the check the definitions of an absolute value and a valuation. We start with v_s. Clearly we have that $v_s(x) = \infty$ if, and only if, $x = 0$. Let $x, y \in K$, if $x = 0$ or $y = 0$, $xy = 0$ and $\infty = v_s(xy) = v(x) + v(y) = \infty$. Assume $x, y \neq 0$, then

$$v_s(xy) = -s \log |xy| = -s \log (|x|\,|y|) = -s \log |x| - s \log |y| = v_s(x) + v_s(y).$$

Let again be $x, y \in K$. If $x = y = 0$, then $\infty = v_s(x+y) = \min\{v_s(x), v_s(y)\} = \infty$. If $x = 0$, $y \neq 0$ (or $y = 0$, $x \neq 0$), $v_s(x+y) = v_s(y) = \min\{v_s(x), v_s(y)\}$ (or $v_s(x+y) = v_s(x) = \min\{v_s(x), v_s(y)\}$). Assume now $x, y \neq 0$. We have

$$\begin{aligned} v_s(x+y) &= -s \log |x+y| \\ &\geq -s \log (\max\{|x|, |y|\}) \\ &= \min\{-s \log |x|, -s \log |y|\} \\ &= \min\{v_s(x), v_s(y)\}. \end{aligned}$$

Therefore v_s is a valuation on K. Assume now $s, s' > 0$, $s \neq s'$. For all $0 \neq x \in K$, we have

$$v_s(x) = -s \log |x| = \left(\frac{s}{s'}\right)(-s' \log |x|) = \frac{s}{s'} v_{s'}(x).$$

This means that v_s and $v_{s'}$ are equivalent. We continue with $|\ \ |_q$. Clearly we have that $|x| = 0$ if, and only if, $x = 0$ and since $q > 1 > 0$ that $|x|_q \geq 0$ for all $x \in K$. Let $x, y \in K$, if $x = 0$ or $y = 0$, $xy = 0$ and $0 = |xy|_q = |x|_q\,|y|_q = 0$. Assume $x, y \neq 0$, then

$$|xy|_q = q^{-v(xy)} = q^{-v(x)-v(y)} = q^{-v(x)} q^{-v(y)} = |x|_q\,|y|_q.$$

Since $\max\left\{|x|_q, |y|_q\right\} \leq |x|_q + |y|_q$, it suffices to show that $|x+y|_q \leq \max\left\{|x|_q, |y|_q\right\}$. Let again be $x, y \in K$. If $x = y = 0$, $0 = |x+y|_q = \max\left\{|x|_q, |y|_q\right\} = 0$. If $x = 0$, $y \neq 0$ (or $y = 0$, $x \neq 0$), $|x+y|_q = |y|_q = \max\left\{|x|_q, |y|_q\right\}$ (or $|x+y|_q = |x|_q = \max\left\{|x|_q, |y|_q\right\}$). Assume now $x, y \neq 0$. We have

$$\begin{aligned} |x+y|_q &= q^{-v(x+y)} \\ &\leq q^{-\min\{v(x), v(y)\}} \\ &= \max\left\{q^{-v(x)}, q^{-v(y)}\right\} \\ &= \max\left\{|x|_q, |y|_q\right\}. \end{aligned}$$

Therefore, $|\ \ |_q$ is a non-Archimedean absolute value on K. Assume now $q, q' > 1$, $q \neq q'$ and set $r := \frac{\log q}{\log q'}$. For all $0 \neq x \in K$, we have

$$|x|_q = q^{-v(x)} = q'^{-rv(x)} = |x|_{q'}^r.$$

Consequently, $|\ \ |_q$ and $|\ \ |_{q'}$ are equivalent.

$\square$

From now on, when we will deal with fields with a non-Archimedean absolute value, according to theorem 2, we will just speak of a field with a valuation.

Remark on terminology: Note that some authors use the term "exponential valuation" rather than "valuation". In this case the term "valuation" means "absolute value".

1.4 Valuation ring and residue field

Theorem 3. *Let K be a field, v be a valuation on K and denote by $|\ \ |$ a corresponding non-Archimedean absolute value. Then:*

1. *the set*
$$\mathcal{O} := \{x \in K \mid v(x) \geq 0\} = \{x \in K \mid |x| \leq 1\}$$
*is an integral domain and a maximal proper subring of K, called the **valuation ring**; moreover, for all $0 \neq x \in K$, we have that $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$,*

2. *the set*
$$\mathcal{O}^* := \{x \in K \mid v(x) = 0\} = \{x \in K \mid |x| = 1\}$$
is the group of units of $\mathcal{O}$,

3. *the set*

$$\mathfrak{p} := \mathcal{O} \setminus \mathcal{O}^* = \{x \in K \mid v(x) > 0\} = \{x \in K \mid |x| < 1\} = \{x \in \mathcal{O} \mid x^{-1} \notin \mathcal{O}\}$$

is the unique maximal ideal of $\mathcal{O}$*.*

A ring that has a unique maximal ideal is called a ***local ring****. By 3.,* $\mathcal{O}$ *is a local ring. Besides, two equivalent valuations (or two equivalent non-Archimedean absolutes values) on* K *give the same valuation ring.*

Proof: We just consider the valuation v on K, since by theorem 2 the valuations and the non-Archimedean absolute values are closely related.
It is easy to check that if two valuation v and v' on K are equivalent, i.e., there is a positive real s such that $v = sv'$, the sets $\mathcal{O}$, $\mathcal{O}^*$ and $\mathfrak{p}$ are the same since if $v(x) \geq 0$, resp. $v(x) = 0$, resp. $v(x) > 0$, $v'(x) = s^{-1}v(x) \geq 0$, resp. $v'(x) = s^{-1}v(x) = 0$, resp. $v'(x) = s^{-1}v(x) > 0$.
To prove that $\mathcal{O}$ is a integral domain, it suffices to show that is closed under addition and multiplication and that every element in $\mathcal{O}$ has an additive inverse in $\mathcal{O}$, since all the remaining properties are verified on the field K and therefore also for the subset $\mathcal{O}$. Clearly, $0, 1 \in \mathcal{O}$ since $v(0) = \infty > v(1) = 0 \geq 0$. Take $0 \neq x \in \mathcal{O}$ and so $-x \in K$. We have $\infty = v(0) = v(x - x) > \min\{v(x), v(-x)\}$ since $x \neq 0$ (and therefore $-x \neq 0$). By property *3.* of Lemma 2, this implies that $v(-x) = v(x) \geq 0$, hence $-x \in \mathcal{O}$. Let $x, y \in \mathcal{O}$. Then $v(xy) = v(x) + v(y) \geq 0$ and so $xy \in \mathcal{O}$. Similarly, $v(x + y) \geq \min\{v(x), v(y)\} \geq 0$ and so $x + y \in \mathcal{O}$. Take now $0 \neq x \in K$. If $v(x) \geq 0$, $x \in \mathcal{O}$, and if $v(x) < 0$, $v(x^{-1}) = -v(x) > 0$ and hence $x^{-1} \in \mathcal{O}$. Let now be $z \in K \setminus \mathcal{O}$. We want to show that $\mathcal{O}[z] = K$ in order to prove that $\mathcal{O}$ is a maximal proper subring of K. Since $z \notin \mathcal{O}$, $z^{-1} \in \mathcal{O}$ and $v(z^{-1}) > 0$. Take an element $y \in K$, then there exists $k \in \mathbb{N}$ such that $v(yz^{-k}) \geq 0$ since $v(z^{-1}) > 0$. Consequently, $w := yz^{-k} \in \mathcal{O}$ and finally $y = wz^k \in \mathcal{O}[z]$ and so $\mathcal{O}[z] = K$.
Assume now $x \in \mathcal{O}$ and $x^{-1} \in \mathcal{O}$. If $v(x) > 0$, $v(x^{-1}) = -v(x) < 0$, therefore $v(x) = 0$. The other inclusion is obvious. This shows that $\mathcal{O}^*$ is the group of units of $\mathcal{O}$.
We show now that $\mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$. Let $x \in \mathfrak{p}$ and $z \in \mathcal{O}$. Then $v(zx) = v(z) + v(x) \geq v(x) > 0$ and hence $zx \in \mathfrak{p}$. Let now $x, y \in \mathfrak{p}$, then $v(x - y) \geq \min\{v(x), v(-y)\}$. If $y = 0$, $x - y \in \mathfrak{p}$. If $y \neq 0$, we have that $\infty = v(0) = v(y - y) > \min\{v(y), v(-y)\}$ and, as before, $v(y) = v(-y) > 0$. Therefore $v(x - y) > 0$. This means that $x - y \in \mathfrak{p}$ and hence $\mathfrak{p}$ is an additive subgroup of $\mathcal{O}$. This shows that $\mathfrak{p}$ is an ideal of $\mathcal{O}$. Assume now there exists an ideal A of $\mathcal{O}$ such that $\mathfrak{p} \subsetneq A$. Then there exists $x \in \mathcal{O}^* \cap A$. Therefore $1 \in A$ and $A = \mathcal{O}$. This shows that $\mathfrak{p}$ is maximal. Assume now that there is a maximal ideal $B \neq \mathcal{O}$ of $\mathcal{O}$ such that $B \neq \mathfrak{p}$. We must have that $B \cap \mathcal{O}^* = \{0\}$, because if not, $1 \in B$ and so $B = \mathcal{O}$. This implies that $B \subsetneq \mathfrak{p}$ and therefore B is not maximal.

□

The theorem tells us that for two equivalent valuation we have the same valuation ring, so, from now on, if a valuation is discrete, we can assume without loss of generality that it is normalized. Clearly if a result holds for a normalized valuation, it holds also for a discrete valuation.

Definition 7. *The field $\mathcal{K} := \mathcal{O}/\mathfrak{p}$ is called the* ***residue field*** *of $\mathcal{O}$.*

We have an interesting property when a valuation is normalized.

Lemma 3. *Let v be a normalized valuation on K, then for all $0 \neq x \in K$ we can write $x = ut^n$, where $t \in \mathfrak{p}$ with $v(t) = 1$, $u \in \mathcal{O}^*$ and $n \in \mathbb{Z}$. An element $x \in \mathfrak{p}$ such that $v(x) = 1$ is called* ***prime element****.*

Proof: Since $v(K^*) = \mathbb{Z}$, there exists an element $t \in K$ with $v(t) = 1$. Therefore $t \in \mathfrak{p}$. Let $0 \neq x \in K$. We have that $v(x) = m$ for some $m \in \mathbb{Z}$. Hence $v(xt^{-m}) = 0$ and so $u := xt^{-m} \in \mathcal{O}^*$ and finally $x = ut^m$.

□

Recall that an ideal of a ring is principal, if it is generated by one element.

Theorem 4. *Assume v is a normalized valuation on K, then $\mathcal{O}$ is a principal ideal domain, i.e., every ideal of $\mathcal{O}$ is principal. Moreover, all the ideals of $\mathcal{O}$ different from $\{0\}$ are of the form*

$$\mathfrak{p}^n = t^n\mathcal{O} = \{x \in K \mid v(x) \geq n\},\ n \geq 0,$$

where $t \in \mathfrak{p}$, $v(t) = 1$. Furthermore, we have

$$\mathfrak{p}^n/\mathfrak{p}^{n+1} \cong \mathcal{K},\ n \geq 1,$$

as K-vector spaces.

Proof: Let $\{0\} \neq A \subseteq \mathcal{O}$ be an ideal of the valuation ring and $0 \neq x \in A$ such that $n := v(x) \leq v(y)$ for all $y \in A$. By lemma 3, $x = ut^n$ for some prime element $t \in \mathfrak{p}$ and some $u \in \mathcal{O}^*$. This implies that $t^n\mathcal{O} \subseteq A$. Take now $y \in A$. According to lemma 3, $y = wt^m$, $w \in \mathcal{O}^*$. Since $y \in A$, $m = v(y) \geq n = v(x)$, so we can write $y = (wt^{m-n})\,t^n \in t^n\mathcal{O}$, hence $A \subseteq t^n\mathcal{O}$.
Consider now the K-homomorphism

$$\begin{array}{rccc} \phi: & \mathfrak{p}^n & \longrightarrow & \mathcal{O}/\mathfrak{p} \\ & at^n & \mapsto & a \mod \mathfrak{p}, \end{array}$$

with $t \in \mathfrak{p}$ a prime element and $a \in \mathcal{O}$. Clearly, ϕ is surjective and its kernel is $\mathfrak{p}^{n+1}$. Hence, ϕ is a K-isomorphism.

□

This theorem gives us a useful lemma.

Lemma 4. *Let v, v' be two discrete valuation on K such that $\mathcal{O}_v = \mathcal{O}_{v'}$, where $\mathcal{O}_v$, resp. $\mathcal{O}_{v'}$, denotes the valuation ring generated by v, resp. by v'. Then v and v' are equivalent.*

Proof: Clearly, $\mathcal{O}_v = \mathcal{O}_{v'}$ implies $\mathcal{O}_v^* = \mathcal{O}_{v'}^*$ and $\mathfrak{p}_v = \mathfrak{p}_{v'}$. By theorem 4, there is an element $t \in \mathfrak{p}_v = \mathfrak{p}_{v'}$ such that $\mathfrak{p}_v = t\mathcal{O}_v = t\mathcal{O}_{v'} = \mathfrak{p}_{v'}$. Again by theorem 4, t must have the minimal valuation among the elements of $\mathfrak{p}_v = \mathfrak{p}_{v'}$. Set $v(t) =: s$ and $v'(t) =: s'$ and this gives $v(K^*) = s\mathbb{Z}$ and $v'(K^*) = s'\mathbb{Z}$. Clearly, for all $x \in \mathcal{O}_v^* = \mathcal{O}_{v'}^*$, $0 = v(x) = v'(x) = 0$. Consider $x \in K \setminus \mathcal{O}_v^* = K \setminus \mathcal{O}_{v'}^*$. Then, by lemma 3, $x = t^n u$ for some $n \in \mathbb{Z}$ and some $u \in \mathcal{O}_v^* = \mathcal{O}_{v'}^*$. So, $v(x) = ns$ and $v'(x) = s'n$ and this gives $v(x) = \frac{s}{s'} v'(x)$. Since x was arbitrary, the result holds for all $x \in K \setminus \mathcal{O}_v^* = K \setminus \mathcal{O}_{v'}^*$. Therefore v and v' are equivalent.

□

1.5 Example: the field of rational numbers $\mathbb{Q}$

In this paragraph, we deal with the field of rational numbers $\mathbb{Q}$.
We know that on $\mathbb{Q}$ we have the restriction of the Archimedean absolute value on $\mathbb{R}$

$$\begin{array}{rccc} |\ \ | : & \mathbb{Q} & \longrightarrow & \mathbb{R} \\ & x & \mapsto & \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} . \end{array}$$

Since $|\ \ |$ is Archimedean, all the constructions that we made above are worthless.
We want a non-Archimedean absolute value on $\mathbb{Q}$. Consider a prime $p > 1$. For all $x \in \mathbb{Q}$ we can write

$$x = p^n \frac{a}{b}$$

with $n \in \mathbb{Z}$, $a \in \mathbb{Z}$, $0 \neq b \in \mathbb{N}$, $(a, b) = 1$, $p \nmid a$ and $p \nmid b$. Define the function $v_p : \mathbb{Q} \to \mathbb{R} \cup \{\infty\}$ as follows:

$$v_p(x) = v_p\left(p^n \frac{a}{b}\right) = n,$$

for all $0 \neq x \in \mathbb{Q}$, and $v_p(0) = \infty$. Clearly v_p is normalized valuation. According to theorem 3, we have the valuation ring

$$\mathcal{O}_p = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, (a, b) = 1, p \nmid b \right\},$$

its group of units

$$\mathcal{O}_p^* = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, (a, b) = 1, p \nmid a, p \nmid b \right\},$$

and its unique maximal ideal

$$\mathfrak{p}_p = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\}, (a,b) = 1, p \mid a, p \nmid b \right\}.$$

Clearly $p \in \mathbb{Q}$ is a prime element and therefore all the non-zero ideals of $\mathcal{O}_p$ are of the form

$$\mathfrak{p}_p^n = p^n \mathcal{O}_p, \; n \geq 0.$$

Let now $x = p^n \frac{a}{b} \in \mathcal{O}_p$, $(a,b) = 1$, $a \in \mathbb{Z}$, $b \in \mathbb{N} \setminus \{0\}$, $p \nmid a$, $p \nmid b$ and $n \in \mathbb{N}$. If $n > 0$, $x \in \mathfrak{p}_p$ and therefore $x \equiv 0 \mod \mathfrak{p}_p$. Suppose $n = 0$, then $x \in \mathcal{O}_p^*$. We now that for every $m \in \mathbb{Z}$, we can write

$$m = \sum_{i=0}^{r} m_i p^i$$

with $r, m_i \in \mathbb{N}$, $0 \leq m_i < p$, $0 \leq i \leq r$. Hence, we have

$$a = \sum_{i=0}^{s} a_i p^i \text{ and } b = \sum_{j=0}^{t} b_j p^j$$

with $s, t, a_i, b_j \in \mathbb{N}$, $0 \leq a_i, b_j < p$, $0 \leq i \leq s$, $0 \leq j \leq t$. This gives

$$\begin{aligned} x &= \frac{a}{b} \\ &= \frac{a_0 + a_1 p + \cdots + a_s p^s}{b} \\ &= \frac{a_0}{b} + \frac{a_1 p + \cdots + a_s p^s}{b} \end{aligned}$$

with

$$\frac{a_1 p + \cdots + a_s p^s}{b} = p \frac{a_1 + \cdots + a_s p^{s-1}}{b} \in \mathfrak{p}_p.$$

Moreover, we have that

$$\frac{a_0}{b} = \frac{a_0}{b_0 + b_1 p + \cdots + b_t p^t} = \frac{a_0}{b_0} - \frac{a_0 b_1 p + a_0 b_2 p^2 + \cdots + a_0 b_t p^t}{b_0^2 + b_0 b_1 p + \cdots + b_0 b_t p^t}$$

and so

$$\frac{a_0 b_1 p + a_0 b_2 p^2 + \cdots + a_0 b_t p^t}{b_0^2 + b_0 b_1 p + \cdots + b_0 b_t p^t} \in \mathfrak{p}_p.$$

This means that

$$x \equiv \frac{a_0}{b_0} \mod \mathfrak{p}_p$$

and finally we get that

$$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \cong \mathcal{K}_p = \mathcal{O}_p/\mathfrak{p}_p.$$

By theorem 2, we know that each valuation v_p gives a corresponding family of equivalent non-Archimedean absolute values. We will denote by $| \;\; |_p$ the absolute value

$$|x|_p := p^{-v_p(x)},$$

for all $0 \neq x \in \mathbb{Q}$.
We conclude this example with an important result.

Theorem 5. *On $\mathbb{Q}$, each non-trivial absolute value is equivalent to an absolute value $|\ |_p$ or $|\ |$.*

Proof: Let $\|\ \|$ be a non-Archimedean absolute value on $\mathbb{Q}$. We have

$$\|n\| = \|1 + \cdots + 1\| \leq \max\{\|1\|, \ldots, \|1\|\} = 1.$$

There exists a prime $p > 1$ such that $\|p\| < 1$. If not, $\|\ \|$ is the trivial valuation. Consider the set

$$A := \{a \in \mathbb{Z} \mid \|a\| < 1\}.$$

Clearly, A is an ideal of $\mathbb{Z}$ and $p\mathbb{Z} \subseteq A \neq \mathbb{Z}$. Since $p\mathbb{Z}$ is a maximal ideal, we have that $A = p\mathbb{Z}$. Take $a \in \mathbb{Z}$, $a = p^m b$ with $p \nmid b$. This implies that $b \notin A$ and therefore $\|b\| = 1$. This gives

$$\|a\| = \|p^m b\| = \|p\|^m = |a|_p^s,$$

with $s := -\frac{\log\|p\|}{\log p}$. Therefore $\|\ \|$ is equivalent to $|\ |_p$.
Assume now that $\|\ \|$ is Archimedean. For all integer $m, n > 1$, we have

$$\|m\|^{\frac{1}{\log m}} = \|n\|^{\frac{1}{\log n}}.$$

Indeed, we can write

$$m = a_0 + a_1 n + \cdots + a_r n^r$$

with $r, a_i \in \mathbb{N}$, $0 \leq a_i < n$, $0 \leq i \leq r$. Clearly $n^r \leq m$, and so $r \leq \frac{\log m}{\log n}$. Moreover,

$$\|a_i\| \leq \|1 + \cdots + 1\| \leq a_i \|1\| \leq n$$

gives that

$$\|m\| \leq \sum_{i=0}^{r} \|a_i\| \|n\|^i \leq \sum_{i=0}^{r} \|a_i\| \|n\|^r \leq \left(1 + \frac{\log m}{\log n}\right) n \|n\|^{\frac{\log m}{\log n}}.$$

Replacing m by m^k, $k \in \mathbb{N}$, and taking the k-th root, we get that

$$\|m\| = \sqrt[k]{\|m^k\|} \leq \sqrt[k]{\left(1 + \frac{k \log m}{\log n}\right) n \|n\|^{\frac{k \log m}{\log n}}} = \|n\|^{\frac{\log m}{\log n}} \sqrt[k]{\left(1 + \frac{k \log m}{\log n}\right) n},$$

and when k goes to ∞ we have

$$\|m\| \leq \|n\|^{\frac{\log m}{\log n}} \text{ and so } \|m\|^{\frac{1}{\log m}} \leq \|n\|^{\frac{1}{\log n}}.$$

Exchanging the roles of m and n we get the inverse inequality.
Since $\|n\|^{\frac{1}{\log n}} > 0$, there exists $s \in \mathbb{R}$ such that $e^s = \|n\|^{\frac{1}{\log n}}$ and so $\|n\| = e^{s \log n}$. Hence, for all $x \in \mathbb{Q}$, $x > 0$, we have

$$\|x\| = e^{s \log x} = x^s = |x|^s$$

Since $\|x\| = \|-x\|$, we have that $\|\ \|$ is equivalent to $|\ |$.

□

1.6 Example: the rational function field $\mathbb{F}_q(T)$

Let $q = p^n$, $p > 1$ a prime and $n \in \mathbb{N} \setminus \{0\}$. The field

$$\mathbb{F}_q(T) = \left\{ \frac{f(T)}{g(T)} \;\middle|\; f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\} \right\}$$

is called **rational function field**.
A polynomial is called **monic** if the leading coefficient is equal to 1. Let $p(T) \in \mathbb{F}_q[T]$ be a monic, irreducible polynomial. For all $f(T) \in \mathbb{F}_q(T)$ we can write

$$f(T) = p(T)^n \frac{g(T)}{h(T)},$$

with $n \in \mathbb{Z}$, $g(T) \in \mathbb{F}_q[T]$ such that $p(T) \nmid g(T)$ and $h(T) \in \mathbb{F}_q[T] \setminus \{0\}$ such that $p(T) \nmid h(T)$.
Define the function $v_{p(T)} : \mathbb{F}_q(T) \to \mathbb{Z} \cup \{\infty\}$ as follows:

$$v_{p(T)}(f(T)) = v_{p(T)} \left(p(T)^n \frac{g(T)}{h(T)} \right) := n,$$

for all $0 \neq f(T) \in \mathbb{F}_q(T)$ and $v_{p(T)}(0) = \infty$. Obviously, $v_{p(T)}$ is a normalized valuation and we have, according to theorem 3, the valuation ring

$$\mathcal{O}_{p(T)} = \left\{ \frac{f(T)}{g(T)} \middle| f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \nmid g(T) \right\},$$

its group of units

$$\mathcal{O}^*_{p(T)} = \left\{ \frac{f(T)}{g(T)} \middle| f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \nmid F(T), p(T) \nmid g(T) \right\},$$

and its unique maximal ideal

$$\mathfrak{p}_{p(T)} = \left\{ \frac{f(T)}{g(T)} \middle| f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \mid f(T), p(T) \nmid g(T) \right\}.$$

Moreover, the residue field of $\mathcal{O}_{p(T)}$, $\mathcal{K}_{p(T)} = \mathcal{O}_{p(T)}/\mathfrak{p}_{p(T)}$ is isomorphic to $\mathbb{F}_q[T]/(p(T))$. Indeed, consider the ring homomorphism

$$\begin{array}{rccc} \varphi : & \mathbb{F}_q[T] & \longrightarrow & \mathcal{O}_{p(T)}/\mathfrak{p}_{p(T)} \\ & f(T) & \mapsto & f(T) \mod \mathfrak{p}_{p(T)}. \end{array}$$

Clearly, the kernel of φ is the ideal $(p(T))$ generated by $p(T)$ in $\mathbb{F}_q[T]$. Take now $h(T) \in \mathcal{O}_{p(T)}$. We can write $h(T) = \frac{r(T)}{s(T)}$ with $r(T), s(T) \in \mathbb{F}_q[T]$, such that $s(T) \neq 0$ and $p(T) \nmid s(T)$. Thus, there exist $a(T), b(T) \in \mathbb{F}_q[T]$ with $a(T)p(T) + b(T)s(T) = 1$ and therefore

$$h(T) = 1 \cdot h(T) = \frac{a(T)r(T)}{s(T)} p(T) + b(T)r(T),$$

and so

$$h(T) \equiv b(T)r(T) \mod \mathfrak{p}_{p(T)}.$$

Since $b(T), r(T) \in \mathbb{F}_q[T]$, φ is surjective and we have an isomorphism

$$\mathbb{F}_q[T]/(p(T)) \cong \mathcal{K}_{p(T)} = \mathcal{O}_{p(T)}/\mathfrak{p}_{p(T)}.$$

Let $f(T) \in \mathbb{F}_q(T)$. We can write

$$f(T) = \frac{g(T)}{h(T)},$$

with $g(T) \in \mathbb{F}_q[T]$, $h(T) \in \mathbb{F}_q[T] \setminus \{0\}$. Consider now the function

$$v_\infty \,:\, \mathbb{F}_q(T) \to \mathbb{Z} \cup \{\infty\}$$

defined as follows:

$$v_\infty(f(T)) = v_\infty\left(\frac{g(T)}{h(T)}\right) := \deg h(T) - \deg g(T),$$

for all $0 \neq f(T) \in \mathbb{F}_q(T)$ and $v_\infty(0) = \infty$. Obviously, v_∞ is a normalized valuation and, by theorem 3, we have the valuation ring

$$\mathcal{O}_\infty = \left\{ \frac{f(T)}{g(T)} \middle| \, f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg f(T) \leq \deg g(T) \right\},$$

its group of units

$$\mathcal{O}_\infty^* = \left\{ \frac{f(T)}{g(T)} \middle| \, f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg f(T) = \deg g(T) \right\},$$

and its unique maximal ideal

$$\mathfrak{p}_\infty = \left\{ \frac{f(T)}{g(T)} \middle| \, f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg f(T) < \deg g(T) \right\}.$$

We have that $T^{-1} \in \mathbb{F}_q(T)$ is a prime element, since $v_\infty(T^{-1}) = \deg T - \deg 1 = 1$. Therefore, all non-zero ideal are of the form

$$\mathfrak{p}_\infty^n = (T)^{-n}\,\mathcal{O}_\infty,\ n \geq 0.$$

Take $f(T) \in \mathcal{O}_\infty$, $f(T) = \frac{g(T)}{h(T)}$, with $g(T) \in \mathbb{F}_q[T]$, $h(T) \in \mathbb{F}_q[T] \setminus \{0\}$ and $n := \deg g(T) \leq \deg h(T) =: m$. Then we can write

$$g(T) = a_nT^n + a_{n-1}T^{n-1} + \cdots + a_0 \text{ and } h(T) = b_mT^m + b_{m-1}T^{m-1} + \cdots + b_0,$$

with $a_i \in \mathbb{F}_q$, $0 \leq i \leq n$, $a_n \neq 0$, and $b_j \in \mathbb{F}_q$, $0 \leq j \leq m$, $b_m \neq 0$. If $n < m$, we have $f(T) \equiv 0 \mod \mathfrak{p}_\infty$. If $n = m$, we have

$$\begin{aligned} f(T) &= \frac{g(T)}{h(T)} \\ &= \frac{a_nT^n + a_{n-1}T^{n-1} + \cdots + a_0}{b_nT^n + b_{n-1}T^{m-1} + \cdots + b_0} \\ &= \frac{a_nT^n}{b_nT^n + b_{n-1}T^{n-1} + \cdots + b_0} + \frac{a_{n-1}T^{n-1} + \cdots + a_0}{b_nT^n + b_{n-1}T^{n-1} + \cdots + b_0}, \end{aligned}$$

with

$$\frac{a_{n-1}T^{n-1}+\cdots+a_0}{b_nT^n+b_{n-1}T^{n-1}+\cdots+b_0}\in\mathfrak{p}_\infty.$$

Moreover

$$\frac{a_nT^n}{b_nT^n+b_{n-1}T^{n-1}+\cdots+b_0}=\frac{a_n}{b_n}-\frac{a_nb_{n-1}T^{n-1}+a_nb_{n-2}T^{n-2}+\cdots+a_nb_0}{b_n^2T^n+b_nb_{n-1}T^{n-1}+\cdots+b_nb_0},$$

with

$$\frac{a_nb_{n-1}T^{n-1}+a_nb_{n-2}T^{n-2}+\cdots+a_nb_0}{b_n^2T^n+b_nb_{n-1}T^{n-1}+\cdots+b_nb_0}\in\mathfrak{p}_\infty.$$

Therefore, we get that

$$f(T)\equiv\frac{a_n}{b_n}\mod\mathfrak{p}_\infty.$$

This means that

$$\mathcal{K}_\infty\cong\mathbb{F}_q.$$

Note that in the case of the above valuations, we can embed the field $\mathbb{F}_q$ in the residue field and this leads to an important

Definition 8. *Let v be a valuation on $\mathbb{F}_q(T)$ and $\mathfrak{p}_v$ the maximal ideal of the valuation ring with respect to v. Then*

$$\deg\mathfrak{p}_v:=[\mathcal{K}_v:\mathbb{F}_q]$$

is called the ***degree*** *of $\mathfrak{p}_v$.*

Note that if a valuation is defined as above by an irreducible, monic polynomial $p(T)\in\mathbb{F}_q[T]$, the degree of $\mathfrak{p}_{p(T)}$ is $\deg\mathfrak{p}_{p(T)}=\deg p(T)$. Indeed $\mathcal{K}_{p(T)}\cong\mathbb{F}_q[T]/(p(T))$ and therefore $\deg\mathfrak{p}_{p(T)}=[\mathcal{K}_{p(T)}:\mathbb{F}_q]=\deg p(T)$. Moreover $\deg\mathfrak{p}_\infty=1$, since $\mathcal{K}_\infty\cong\mathbb{F}_q$.

We have an interesting lemma about the degree.

Lemma 5. *Let v be a valuation on $\mathbb{F}_q(T)$ and $0\neq f(T)\in\mathfrak{p}_v$. Then we have*

$$\deg\mathfrak{p}_v\leq[\mathbb{F}_q(T):\mathbb{F}_q(f(T))]<\infty,$$

where $\mathbb{F}_q(f(T))$ denotes the field generated by $f(T)$ over $\mathbb{F}_q$.

Proof: Note that $\mathbb{F}_q\subset\mathbb{F}_q(f(T))$. Write $f(T)=\frac{g(T)}{h(T)}$, with $g(T),h(T)\in\mathbb{F}_q[T]$. Then, $p(X):=f(T)h(X)-g(X)$ is a polynomial in X with coefficients in $\mathbb{F}_q(f(T))$. Clearly T is a root of $p(X)$, hence $[\mathbb{F}_q(T):\mathbb{F}_q(f(T))]<\infty$.
For the remaining inequality, it suffices to show that any $f_1(T),\ldots,f_n(T)\in\mathcal{O}_v$, whose residue class in $\bar{f}_1(T),\ldots,\bar{f}_n(T)\in\mathcal{K}_v$ are linearly independent over $\mathbb{F}_q$, are linearly independent over $\mathbb{F}_q(f(T))$. Assume that

$$\sum_{i=0}^n\varphi_if_i(T)=0$$

for some $\varphi_i \in \mathbb{F}_q(f(T))$, $0 \leq i \leq n$, not all zero. In fact, φ_i are linear combinations of powers of $f(T)$ with coefficients on $\mathbb{F}_q$. W.l.o.g. we can assume that φ_i are polynomials in $f(T)$ and not all of the φ_i are divisible by $f(T)$, so we can write $\varphi_i = a_i + f(T)g_i$, with $a_i \in \mathbb{F}_q$ not all zero, $g_i \in \mathbb{F}_q[f(T)]$, $0 \leq i \leq n$. Since $f(T) \in \mathfrak{p}_v$ and $g_i \in \mathbb{F}_q[f(T)] \subseteq \mathcal{O}_v$, we have that $\varphi_i \equiv a_i \mod \mathfrak{p}_v$, for all $0 \leq i \leq n$. Hence

$$0 \equiv \sum_{i=0}^{n} \varphi_i f_i(T) \equiv \sum_{i=0}^{n} a_i \bar{f}_i(T) \mod \mathfrak{p}_v.$$

This contradicts the linear independence of $\bar{f}_1(T), \ldots, \bar{f}_n(T) \in \mathcal{K}_v$ over $\mathbb{F}_q$.

□

We can state now a very important theorem concerning the rational function field $\mathbb{F}_q(T)$.

Theorem 6. *All the non-trivial valuations on the rational function field $\mathbb{F}_q(T)$ are equivalent to a valuation $v_{p(T)}$, for $p(T) \in \mathbb{F}_q[T]$ a monic, irreducible polynomial, or to v_∞.*

Proof: By lemma 4, it suffices to show that if v is a non-trivial valuation different from v_∞, there is an irreducible, monic polynomial $p(T) \in \mathbb{F}_q[T]$ such that $\mathcal{O}_{p(T)} = \mathcal{O}_v$.
Assume first that $T \in \mathcal{O}_v$, then, obviously, $\mathbb{F}_q[T] \subseteq \mathcal{O}_v$. Set $I := \mathbb{F}_q[T] \cap \mathfrak{p}_v$; this is a prime ideal of $\mathbb{F}_q[T]$. Indeed $\mathfrak{p}_v$ is a maximal ideal, therefore is prime, and $\mathbb{F}_q[T]$ is a ring. The map $\mathbb{F}_q[T] \to \mathcal{K}_v$ induces an embedding $\mathbb{F}_q[T]/I \hookrightarrow \mathcal{K}_v$, and therefore, by lemma 5, $I \neq \{0\}$. Indeed, if $I = \{0\}$, we have that $\mathbb{F}_q[T]/I = \mathbb{F}_q[T]$ and so $\mathbb{F}_q[T] \hookrightarrow \mathcal{K}_v$. But $\infty = [\mathbb{F}_q[T] : \mathbb{F}_q] \leq [\mathcal{K}_v : \mathbb{F}_q] < \infty$. Since $I \neq \{0\}$ and I is prime, there is a unique irreducible, monic polynomial $p(T) \in \mathbb{F}_q[T]$ such that $I = p(T)\mathbb{F}_q[T]$ (recall that the ring $\mathbb{F}_q[T]$ is principal). Any $g(T) \in \mathbb{F}_q[T]$ with $p(T) \nmid g(T)$ is not in I, so $g(T) \notin \mathfrak{p}_v \subset \mathcal{O}_v$, therefore $g(T)^{-1} \in \mathcal{O}_v$. Hence, we have

$$\mathcal{O}_{p(T)} = \left\{ \frac{f(T)}{g(T)} \middle| \, f(T) \in \mathbb{F}_q[T], g(T) \in \mathbb{F}_q[T] \setminus \{0\}, p(T) \nmid g(T) \right\} \subseteq \mathcal{O}_v.$$

By theorem 3, all valuation rings are maximal proper subrings of $\mathbb{F}_q(T)$, therefore $\mathcal{O}_{p(T)} = \mathcal{O}_v$. Therefore v and $v_{p(T)}$ are equivalent.
Assume now that $T \notin \mathcal{O}_v$. We have that $\mathbb{F}_q\left[T^{-1}\right] \subseteq \mathcal{O}_v$, $T^{-1} \in \mathfrak{p}_v \cap \mathbb{F}_q\left[T^{-1}\right]$ and clearly $\mathfrak{p}_v \cap \mathbb{F}_q\left[T^{-1}\right] = T^{-1}\mathbb{F}_q\left[T^{-1}\right]$, since $\mathfrak{p}_v \cap \mathbb{F}_q\left[T^{-1}\right]$ is a prime ideal. As before, if $g\left(T^{-1}\right) \in \mathbb{F}_q\left[T^{-1}\right]$ with $T^{-1} \nmid g\left(T^{-1}\right)$, $g\left(T^{-1}\right) \notin \mathfrak{p}_v$ and so $g\left(T^{-1}\right)^{-1} \in \mathcal{O}_v$. This

gives

$$\begin{aligned}
\mathcal{O}_v &\supseteq \left\{ \frac{f\left(T^{-1}\right)}{g\left(T^{-1}\right)} \middle| f\left(T^{-1}\right) \in \mathbb{F}_q\left[T^{-1}\right], g\left(T^{-1}\right) \in \mathbb{F}_q\left[T^{-1}\right] \setminus \{0\}, T^{-1} \nmid g\left(T^{-1}\right) \right\} \\
&= \left\{ \frac{a_0 + a_1 T^{-1} + \cdots + a_n T^{-n}}{b_0 + b_1 T^{-1} + \cdots + b_m T^{-m}} \middle| b_0 \neq 0 \right\} \\
&= \left\{ \frac{a_0 T^{m+n} + a_1 T^{m+n-1} + \cdots + a_n T^{m}}{b_0 T^{m+n} + b_1 T^{m+n-1} + \cdots + b_m T^{n}} \middle| b_0 \neq 0 \right\} \\
&= \left\{ \frac{u(T)}{v(T)} \middle| u(T) \in \mathbb{F}_q[T], v(T) \in \mathbb{F}_q[T] \setminus \{0\}, \deg u(T) \leq \deg v(T) \right\} \\
&= \mathcal{O}_\infty .
\end{aligned}$$

According to theorem 3, this implies that $\mathcal{O}_v = \mathcal{O}_\infty$ and so v is equivalent to v_∞.

□

This theorem and the theorem 5 for $\mathbb{Q}$ are very similar. The valuations v_p, given by a prime p, on $\mathbb{Q}$ correspond to the valuations $v_{p(T)}$, given by a monic, irreducible polynomial $p(T)$, on $\mathbb{F}_q(T)$ and we have a difference between the usual absolute value $|\ |$ on $\mathbb{Q}$ and the valuation v_∞ on $\mathbb{F}_q(T)$. The problem is that $|\ |$ is Archimedean, hence we haven't any corresponding valuation.

We conclude this example with a corollary.

Corollary. *There is a bijection between the equivalence classes of non-trivial valuations v on $\mathbb{F}_q(T)$ with $\deg \mathfrak{p}_v = 1$ and $\mathbb{F}_q \cup \{\infty\}$.*

Proof: By theorem 6, the valuations $v_{p(T)}$, $p(T)$ an irreducible, monic polynomial in $\mathbb{F}_q[T]$, and v_∞ represent all the equivalence classes of non-trivial valuations on $\mathbb{F}_q(T)$. Moreover, by the note after the definition 8, the valuations with degree 1 are exactly v_∞ and $v_{T-\alpha}$, for all $\alpha \in \mathbb{F}_q$. Therefore, we have a bijection between $\mathbb{F}_q \cup \{\infty\}$ and the equivalence classes of non-trivial valuations on $\mathbb{F}_q(T)$.

□

Remark: In fact all the results that holds for $\mathbb{F}_q(T)$, holds also for all fields

$$K(T) := \left\{ \frac{f(T)}{g(T)} \middle| f(T) \in K[T], g(T) \in K[T] \setminus \{0\} \right\},$$

where $K[T]$ denotes the ring of polynomials over a field K. $K(T)$ is called **rational function field**.

2 Completion

2.1 Definitions and results

We begin this section with some basic definitions.

Definition 9. *Let K be a field and $|\ |$ an absolute value on K. A sequence $\{a_n\}_{n\in\mathbb{N}}$ in K is called a* ***Cauchy sequence*** *if, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that*

$$n, m \geq N \Longrightarrow |a_n - a_m| < \varepsilon.$$

Another important

Definition 10. *A field K with an absolute value $|\ |$ is called* ***complete*** *if any Cauchy sequence $\{a_n\}_{n\in\mathbb{N}}$ in K converges to an element $a \in K$, i.e.*

$$\lim_{n\to\infty} |a_n - a| = 0.$$

A useful lemma for non-Archimedean absolute values.

Lemma 6. *Let K be a complete field and $|\ |$ a non-Archimedean absolute value on K. Then, for $\{a_n\}_{n\in\mathbb{N}} \subset K$, we have:*

1. *the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if, and only if, $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$,*
2. *the series $\sum_{n=0}^{\infty} a_n$ converges if, and only if, $\lim_{n\to\infty} a_n = 0$,*
3. *Suppose that $\lim_{n\to\infty} a_n = a \neq 0$, then there exists a positive integer N such that for all $m \geq N$, $|a_m| = |a_N| = |a|$.*

Proof:

1. Assume that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_m - a_n| < \varepsilon$, therefore we have also, for all $n \geq N$, $|a_{n+1} - a_n| < \varepsilon$. Conversely, assume that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_{n+1} - a_n| < \varepsilon$. Then, for all $r, s \geq N$,

$$\begin{aligned} |a_r - a_s| &= \left|\textstyle\sum_{i=n}^{m-1}(a_{i+1} - a_i)\right| \\ &\leq \max_{n\leq i<m}\{|a_{i+1} - a_i|\} < \varepsilon. \end{aligned}$$

   Therefore, $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

2. We have that $|a_n| = \left|\sum_{i=0}^{n} a_i - \sum_{j=0}^{n-1} a_j\right|$. Using *1.*, the equivalence is obvious.

3. Since $\lim_{n\to\infty} a_n = a \neq 0$, there exists a positive integer n_a such that, for all $n \geq n_a$, $|a_n - a| < |a|$. Then, using the note we made after definition 3, we have that $|a_n| = |a_n - a + a| = \max\{|a_n - a|, |a|\} = |a|$.

□

Note that for a non-Archimedean absolute value we have the equivalence

$$\exists a > 0 \text{ such that } |x| < a \Leftrightarrow \exists b \in \mathbb{R} \text{ such that } v(x) > b,$$

with v a corresponding valuation. Indeed, for $s > 0$, if $|x| < a$, $v(x) = -s\log|x| > -s\log a$. Conversely, for $q > 1$, if $v(x) > b$, $|x| = q^{-v(x)} < q^{-b}$.

Theorem 7. *Let K be a field and $|\ \ |$ be an absolute value on K. Then, there exists a unique, up to K-isomorphism, complete field $\widehat{K}$ with an absolute value $|\ \ |_{\widehat{K}}$ such that K is embedded in $\widehat{K}$ as a dense subfield and the absolute value on K is a restriction of the absolute value on $\widehat{K}$, i.e., $|x|_{\widehat{K}} = |x|$ if $x \in K$.*

Sketch of the proof: We will not prove this theorem in detail, not because it is too difficult, but because we would need to prove a lot of uninteresting little claims, that can be easily proved by the reader.
Let R be the set of all the Cauchy sequences in K with respect to $|\ \ |$. Define the addition and the multiplication as follows, for all $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \in R$:

$$\{a_n\}_{n\in\mathbb{N}} + \{b_n\}_{n\in\mathbb{N}} := \{a_n + b_n\}_{n\in\mathbb{N}} \text{ and } \{a_n\}_{n\in\mathbb{N}} \cdot \{b_n\}_{n\in\mathbb{N}} := \{a_n b_n\}_{n\in\mathbb{N}}.$$

Indeed R is a ring. Let $\mathfrak{m} \subset R$ be the set of all the Cauchy sequences that converge to 0. It not difficult to prove that $\mathfrak{m}$ is a maximal ideal of R. Now set

$$\widehat{K} := R/\mathfrak{m}.$$

Clearly, $\widehat{K}$ is a field. We have an injection $K \hookrightarrow \widehat{K}$ by sending $a \in K$ to the equivalence class of the Cauchy sequence $(a, a, a, \ldots)$. Hence, we can write $K \subset \widehat{K}$. Take $a \in \widehat{K}$ and let $\{a_n\}_{n\in\mathbb{N}} \in R$ be a representative of a. Then, we have that the sequence $\{|a_n|\}_{n\in\mathbb{N}}$ converges in $\mathbb{R}$, because it is a Cauchy sequence, since $||a_n| - |a_m|| \leq |a_n - a_m|$, by *4.* of lemma 1. Set

$$|a|_{\widehat{K}} := \lim_{n\to\infty} |a_n|,$$

then $|\ \ |_{\widehat{K}}$ is an absolute value on $\widehat{K}$ and, if $a \in K$, we have $|a|_{\widehat{K}} = |a|$. Furthermore,

$$\lim_{n\to\infty} a_n = a$$

in $\widehat{K}$, therefore K is dense in $\widehat{K}$ and $\widehat{K}$ is complete with respect to $|\ \ |_{\widehat{K}}$.
Let $\widehat{K}'$ be another complete field, with respect to an absolute value $|\ \ |_{\widehat{K}'}$, such that K is dense in $\widehat{K}'$ and, for all $x \in K$, $|x|_{\widehat{K}'} = |x|$. Take $a \in \widehat{K}$ and let $\{a_n\}_{n\in\mathbb{N}} \subset K$ be a representative of a. Then, in $\widehat{K}'$, this Cauchy sequence converges to an element $a' \in \widehat{K}'$, because K is dense in $\widehat{K}'$. Define the function $\sigma : \widehat{K} \to \widehat{K}'$ by $\sigma(a) := a'$. It is easy to verify that σ is a K-isomorphism. Furthermore, $|a|_{\widehat{K}} = |\sigma(a)|_{\widehat{K}'}$ because

$$|a|_{\widehat{K}} = \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |a_n|_{\widehat{K}'} = |a|_{\widehat{K}'}.$$

Definition 11. *The field $\widehat{K}$ is called the* ***completion*** *of K.*

Now we look at an Archimedean absolute value on a field K. The following theorem is due to Alexander Ostrowski (1893-1986). We will not prove this theorem.

Theorem 8. *Let K be a complete field with respect to an Archimedean absolute value $|\ |_K$, then there is an isomorphism σ from K to $\mathbb{R}$ or $\mathbb{C}$ such that $|x|_K = |\sigma(x)|^s$ for all $x \in K$ and a fixed $0 \leq s \leq 1$, where $|\ \ |$ denotes the usual absolute value of $\mathbb{R}$ or $\mathbb{C}$.*

Ostrowski's theorem tells us that all complete fields with respect to an Archimedean absolute value are isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Therefore, the completion of a field with an Archimedean absolute value is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.
Let us look now at the completion of a field with a non-Archimedean absolute value. As seen before, there is a valuation v corresponding to the non-Archimedean absolute value on K. In this case, we denote $\widehat{v}$ the valuation of the completion $\widehat{K}$ of K. Clearly, if v is discrete, resp. normalized, $\widehat{v}$ is also discrete, resp. normalized.

Theorem 9. *Let K be a field, $\widehat{K}$ its completion with respect to the valuation v on K. Denote $\widehat{v}$ the corresponding valuation on $\widehat{K}$, $\mathcal{O}$, resp. $\widehat{\mathcal{O}}$ the valuation ring of K, resp. $\widehat{K}$, $\mathfrak{p}$, resp. $\hat{\mathfrak{p}}$, the maximal ideal of $\mathcal{O}$, resp. $\widehat{\mathcal{O}}$ and $\mathcal{K}$, resp. $\widehat{\mathcal{K}}$, the residue field of $\mathcal{O}$, resp. $\widehat{\mathcal{O}}$. Then*

$$\mathcal{K} \cong \widehat{\mathcal{K}}$$

and, if v is discrete,

$$\mathcal{O}/\mathfrak{p}^n \cong \widehat{\mathcal{O}}/\hat{\mathfrak{p}}^n,\ n \geq 1.$$

Proof: By theorem 7, we have $K \subset \widehat{K}$, $\mathcal{O} \subset \widehat{\mathcal{O}}$ and $\mathfrak{p} \subset \hat{\mathfrak{p}}$. The inclusion $\mathcal{O} \subset \widehat{\mathcal{O}}$ gives a homomorphism

$$\varphi\ :\ \mathcal{O} \to \widehat{\mathcal{O}}/\hat{\mathfrak{p}},$$

whose kernel is obviously $\mathfrak{p}$. Let now $x \in \widehat{\mathcal{O}}$. By theorem 7, K is dense in $\widehat{K}$, therefore there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset K$, which converges to $x \in \widehat{K}$. Since $\hat{v}(x) \geq 0$, by lemma 6, there exists a positive integer N such that $\hat{v}(x_n) = \hat{v}(x)$, for all $n \geq N$. Hence, we can assume that $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{O}$. By definition, for all $\xi \in \mathbb{R}$, there is $N \in \mathbb{N}$ such that, for all $n \geq N$, $\hat{v}(x - x_n) > \xi$. Take $\xi > 0$, then $x - x_n \in \hat{\mathfrak{p}}$, and we get $x \equiv x_n \mod \hat{\mathfrak{p}}$. This means that φ is surjective and therefore we have an isomorphism

$$\mathcal{O}/\mathfrak{p} \cong \widehat{\mathcal{O}}/\hat{\mathfrak{p}}.$$

Moreover, if v is discrete, $\hat{v}$ is discrete and all the ideal of $\mathcal{O}$, resp. $\widehat{\mathcal{O}}$, are of the form $\mathfrak{p}^n$, resp. $\hat{\mathfrak{p}}^n$, $n \geq 1$. So we have a homomorphism $\lambda\ :\ \mathcal{O} \to \widehat{\mathcal{O}}/\hat{\mathfrak{p}}^n$, whose kernel is $\mathfrak{p}^n$. By the same argument as above, for all $x \in \widehat{\mathcal{O}}$, there is an element $y_n \in \mathcal{O}$ such that $\hat{v}(x - y_n) \geq n$, for all $n \geq 1$. Therefore $x \equiv y_n \mod \hat{\mathfrak{p}}^n$. Hence, λ is surjective and we have an isomorphism $\mathcal{O}/\mathfrak{p}^n \cong \widehat{\mathcal{O}}/\hat{\mathfrak{p}}^n$.

□

Theorem 10. *Take the same assumption as in the preceding theorem and assume that v is normalized. Let $R \subseteq \mathcal{O}$ be a set of representatives of $\mathcal{K}$ such that $0 \in R$ and let $t \in \mathfrak{p}$ be a prime element. Then we can represent all $x \in \widehat{K}^*$ as a converging series*

$$x = t^m(a_0 + a_1t + a_2t^2 + \dots)$$

with $a_i \in R$, $i \in \mathbb{N}$, $a_0 \neq 0$ and $m \in \mathbb{Z}$.

Proof: Since $\mathfrak{p} \subset \widehat{\mathfrak{p}}$, $t \in \widehat{\mathfrak{p}}$ and $1 = v(t) = \hat{v}(t)$, according to theorem 7. From now on, in this proof, we will use an absolute value corresponding to the valuation $\hat{v}$. By lemma 3, we have that $x = ut^m$, $u \in \widehat{\mathcal{O}}^*$. Since $\mathcal{O}/\mathfrak{p} \cong \widehat{\mathcal{O}}/\hat{\mathfrak{p}}$, $u \mod \hat{\mathfrak{p}}$ has a representative $0 \neq a_0 \in R$ and therefore we can write $u = a_0 + tb_1$ with $b_1 \in \widehat{\mathcal{O}}$. By the same argument, we find also $a_1, a_2, \dots, a_{n-1} \in R$ such that

$$u = a_0 + a_1t + \dots + a_{n-1}t^{n-1} + t^nb_n$$

with $b_n \in \widehat{\mathcal{O}}$. As before, there is an $a_n \in R$ such that $b_n = a_n + tb_{n+1}$, $b_{n+1} \in \widehat{\mathcal{O}}$. Hence,

$$u = a_0 + a_1t + \dots + a_{n-1}t^{n-1} + a_nt^n + t^{n+1}b_{n+1}.$$

We can do this for all $n \in \mathbb{N}$, therefore we have a series

$$\sum_{r=0}^{\infty} a_rt^r.$$

It remains to show that this series converges to u. For all $n \in \mathbb{N}$, we have

$$\begin{aligned} \hat{v}\left(u - \textstyle\sum_{i=0}^n a_it^i\right) &= \hat{v}\left(t^{n+1}b_{n+1}\right) \\ &= \hat{v}\left(t^{n+1}\right) + \hat{v}(b_{n+1}) \\ &= n + 1 + \hat{v}(b_{n+1}) \\ &\geq n + 1, \end{aligned}$$

since $b_{n+1} \in \widehat{\mathcal{O}}$. This gives

$$\lim_{n\to\infty} \hat{v}\left(u - \sum_{i=0}^{n} a_it^i\right) = \infty$$

and hence the series converges to u. Finally, we can write

$$x = ut^m = t^m(a_0 + a_1t + a_2t^2 + \dots)$$

□

We will state some results, without proof, concerning polynomials.
Let K be a complete field with respect to the valuation v. We can extend v to the ring $K[x]$ of the polynomials in one variable over K as follows:

$$v(f) := \min\left\{v(a_0), \dots, v(a_n)\right\},$$

where $f(x) = a_0 + a_1x + \dots + a_nx^n$, $a_i \in K$, $0 \leq i \leq n$, $a_n \neq 0$. A polynomial $f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathcal{O}[x]$ is called **primitive** if $v(f) = 0$, i.e., $f(x) \not\equiv 0 \mod \mathfrak{p}$. The following lemma is due to Kurt Hensel (1861-1941).

Lemma 7. *Let $f(x) \in \mathcal{O}[x]$ be a primitive polynomial. Assume that*

$$f(x) \equiv \bar{g}(x)\bar{h}(x) \mod \mathfrak{p},$$

with $\bar{g}(x), \bar{h}(x) \in \mathcal{K}[x]$. Then, there exists two polynomials $g(x), h(x) \in \mathcal{O}[x]$ with $\deg g(x) = \deg \bar{g}(x)$ *and*

$$g(x) \equiv \bar{g}(x) \mod \mathfrak{p} \text{ and } h(x) \equiv \bar{h}(x) \mod \mathfrak{p},$$

such that

$$f(x) = g(x)h(x).$$

We have an immediate consequence.

Corollary. *For all irreducible polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$, we have*

$$v(f) = \min \{v(a_0), v(a_n)\}.$$

Moreover, if $a_n = 1$ and $a_0 \in \mathcal{O}$, $f \in \mathcal{O}[x]$.

2.2 Example: the field of p-adic numbers $\mathbb{Q}_p$

In this example we deal, as in paragraph 1.5, with the field of rational numbers $\mathbb{Q}$. By theorem 5, we know that the equivalence classes of absolute values on $\mathbb{Q}$ are represented by $| \ |_p$, $p > 1$ prime, and $| \ |$. Theorem 8 tells us that the completion of $\mathbb{Q}$ with respect to $| \ |$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, since $| \ |$ is Archimedean. In fact, we know that this completion is $\mathbb{R}$ (one way to define $\mathbb{R}$ is to complete $\mathbb{Q}$ with respect to the usual absolute value).
We are more interested in non-Archimedean absolute values. Hence, let $p > 1$ be a prime number. The completion of $\mathbb{Q}$ with respect to $| \ |_p$ is denoted $\mathbb{Q}_p$, and called the **field of p-adic numbers**. Instead of $| \ |_p$, we use the corresponding valuation v_p; we use the notation v_p also for the extension of v_p in $\mathbb{Q}_p$. We know that $\mathcal{K}_p \cong \mathbb{Z}/p\mathbb{Z}$, therefore we can take $\{0, \ldots, p-1\}$ as set of representatives of $\mathcal{K}_p$; furthermore, p is a prime element. According to theorem 10, for all $0 \neq x \in \mathbb{Q}_p$, we have

$$x = p^m(a_0 + a_1 p + a_2 p^2 + \cdots) = \sum_{i=m}^{\infty} a_i p^i,$$

with $a_i \in \{0, \ldots, p-1\}$, $i \in \mathbb{N}$, $a_0 \neq 0$ and $m \in \mathbb{Z}$. By the construction we made in the proof of theorem 10, we know that

$$u := a_0 + a_1 p + a_2 p^2 + \cdots = \sum_{i=0}^{\infty} a_i p^i$$

is a unit, i.e., $v_p(u) = 0$. This means that $v_p(x) = m$. Therefore, the valuation ring of $\mathbb{Q}_p$ is

$$\mathbb{Z}_p := \left\{ \sum_{i=m}^{\infty} a_i p^i \,\middle|\, a_i \in \{0, \ldots, p-1\}, a_0 \neq 0, m \geq 0 \right\},$$

called the **ring of p-adic integers**. Its group of units is

$$\mathbb{Z}_p^* = \left\{ \sum_{i=m}^{\infty} \,\middle|\, a_i \in \{0, \dots, p-1\}, a_0 \neq 0, m = 0 \right\}$$

and the unique maximal ideal is $p\mathbb{Z}_p$. Moreover, the residue field of $\mathbb{Z}_p$ is $\mathbb{Z}/p\mathbb{Z}$ since it is isomorphic to the residue field of $\mathcal{O}_p$.

Remark on notation: Note that sometimes in topology the notation $\mathbb{Z}_p$ stands for the finite field $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

2.3 Example: the field of Laurent series $\mathbb{F}_q\left(\left(T^{-1}\right)\right)$

As before, we continue the example of section 1, paragraph 1.6. Let $q = p^n$, $p > 1$ prime, $n \in \mathbb{N} \setminus \{0\}$ and let $\mathbb{F}_q(T)$ be the rational function field. By theorem 6, we know that all equivalence classes of valuation on $\mathbb{F}_q(T)$ are represented by $v_{p(T)}$, $p(T) \in \mathbb{F}_q[T]$ a monic, irreducible polynomial, and v_∞.
Now, we are going to see what happens when we complete $\mathbb{F}_q(T)$ with respect to the absolute value corresponding to v_∞. We know that T^{-1} is a prime element for v_∞ and that the residue field $\mathcal{K}_\infty$ of the valuation ring $\mathcal{O}_\infty$ is isomorphic to $\mathbb{F}_q$, then, by theorem 10, we can write all element $f \neq 0$ of the completion in the form

$$f = \left(T^{-1}\right)^m \left(a_0 + a_1 T^{-1} + a_2 \left(T^{-1}\right)^2 + \dots\right) = \left(T^{-1}\right)^m \sum_{i=0}^{\infty} a_i \left(T^{-1}\right)^i$$

with $a_i \in \mathbb{F}_q$, $i \in \mathbb{N}$, $a_0 \neq 0$ and $m \in \mathbb{Z}$. In fact, we abuse of notations and we write

$$f = f(T) = \sum_{i=-\infty}^{-m} a_i T^i, \; a_{-m} \neq 0.$$

We note the completion of $\mathbb{F}_q(T)$ with respect to v_∞ by $\mathbb{F}_q\left(\left(T^{-1}\right)\right)$. Note the analogy between $f(T) \in \mathbb{F}_q\left(\left(T^{-1}\right)\right)$ and a Laurent series in $\mathbb{C}$. Recall that a Laurent series in $\mathbb{C}$ is a series that allows infinite negative terms and converges in an annulus. We call $\mathbb{F}_q\left(\left(T^{-1}\right)\right)$ the **field of formal Laurent series in T^{-1} over $\mathbb{F}_q$**. As above for $\mathbb{Q}_p$, we use the same notation for the valuation in the completion as in the rational function field. Clearly, we have $v_\infty(f(T)) = m$. The valuation ring of the field of formal Laurent series is the ring

$$\left\{ \sum_{i=-\infty}^{-m} a_i T^i \,\middle|\, a_i \in \mathbb{F}_q, a_{-m} \neq 0, m \geq 0 \right\}$$

and the units are element of the form

$$\sum_{i=-\infty}^{0} a_i T^i,$$

with $a_0 \neq 0$. The unique maximal ideal of the valuation ring is the set of all the series with only negative powers of T. Note that the valuation ring coincides with the ring of formal power series in T^{-1}, since no negative powers of T^{-1} occur.

Contents

References

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