Seminar Darstellungstheorie von Köchern

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Köcher mit Relationen

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In the following A denotes a finite dimensional algebra over an algebraic closed field K.

Definition 1 *An element $a \in A$ is called* ***idempotent*** *if $e^2 = e$. The idempotents $e_1, e_2 \in A$ are called* ***orthogonal*** *if $e_1e_2 = e_2e_1 = 0$. The idempotent e is called* ***primitive*** *if e can't be written as a sum $e = e_1 + e_2$, where e_1 and e_2 are non-zero orthogonal idempotents of A.*
A decomposition $A = Ae_1 \oplus ... \oplus Ae_n$ where $e_1, ..., e_n$ are primitive pairwise orthogonal idempotents of A such that $e_1 + ... + e_n = 1$ is called a ***indecomposable decomposition*** *of A and such a set $\{e_1, ..., e_n\}$* ***complete set of primitive orthogonal idempotents*** *of A.*

Corollary 1 *Let Q be a finite quiver. The element $1 = \sum_{a \in Q_0} \varepsilon_a$ is the identity of KQ and the set $\{\varepsilon_a : a \in Q_0\}$ of all stationary paths is a complete set of primitive orthogonal idempotents for KQ.*

Definition 2 *Let Q be a finite and connected quiver. The two-sided ideal of the path algebra KQ generated (as an ideal) by the arrows of Q is called the* ***arrow ideal*** *of KQ and is denoted by R_Q.*
Furthermore we define $R_Q^l = \bigoplus_{m \geq l} KQ_m$, where KQ_l is the subspace of KQ generated by the set Q_l of all paths in Q of length $l \geq 1$. So R_Q^l is the ideal of KQ generated (as a k-vector space) by the set of all paths of lengths $\geq l$.

Now we can define the notion of admissible ideals.

Definition 3 *Let Q be a finite quiver and R_Q be the arrow ideal of the path algebra KQ. A two-sided ideal $\mathcal{I}$ of KQ is said to be* ***admissible*** *if $\exists m \geq 2$ such that $R_Q^m \subseteq \mathcal{I} \subseteq R_Q^2$.*
If $\mathcal{I}$ is an admissible ideal of KQ, the pair $(Q, \mathcal{I})$ is said to be a ***bound quiver****. The quotient $KQ/\mathcal{I}$ is said to be the algebra of the bound quiver $(Q, \mathcal{I})$ or, simply, a* ***bound quiver algebra****.*

Corollary 2 *Let Q be a finite quiver and $\mathcal{I}$ be an admissible ideal of KQ. The set $\{\varepsilon_a + \mathcal{I} : a \in Q_0\}$ is a complete set of primitive orthogonal idempotents of the bound quiver algebra $KQ/\mathcal{I}$.*

Definition 4 *Let Q be a quiver. A* ***relation*** *in Q with coefficients in K is a K-linear combination of paths of length at least two having the same source and target, i.e a relation ρ is an element of KQ such that $\rho = \sum_{i=1}^{m} \lambda_i w_i$, where the w_i are paths in Q of length at least 2 such that, if $i \neq j$, then the source (or the target, resp.) of w_i coincides with that of w_j .*

Now we formulate our first main theorem:

Theorem 1 *Let $A = KQ/\mathcal{I}$, where Q is a finite connected quiver and $\mathcal{I}$ is an admissible ideal of KQ. There exists a K-linear equivalence of categories*

$$F : ModA \longrightarrow Rep_K(Q, \mathcal{I})$$

that restricts to an equivalence of categories $F : modA \longrightarrow rep_K(Q, \mathcal{I})$

Here we want to sketch the proof in some steps:

- Construction of a functor $F : ModA \rightarrow Rep_K(Q, \mathcal{I})$
- Construction of a K-linear functor $F : Rep_K(Q, \mathcal{I}) \rightarrow ModA$
- It restricts to an equivalence of categories $F : modA \longrightarrow rep_K(Q, \mathcal{I})$

Definition 5 *An idempotent element $e \in A$ is called central, if $ea = ae\ \forall a \in A$.*

Corollary 3 *The decomposition $A = Ae \oplus A(1-e)$ for a central idempotent e is not only a left-A module direct sum, but also an algebra direct product; meaning that A equals the product algebra of Ae and $A(1-e)$ where the multiplication is given point wise. For the category $modA$ we obtain the equivalence $modA \cong modAe \oplus modA(1-e)$.*

Definition 6 *An algebra A is called connected, if A is not a direct product of two algebras, or equivalent, if 0 and 1 are the only central idempotents of A.*

Definition 7 *Assume that A is a K-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. The algebra is called basic if $Ae_i \ncong Ae_j$, for all $i \neq j$. A* ***basic algebra*** *associated to A is the algebra*

$$A^b = e_A A e_A,$$

where $e_A = e_{j_1} + \cdots + e_{j_a}$, and $e_{j_1}, \ldots, e_{j_a}$ are chosen such that $Ae_{j_\alpha} \ncong A_{j_\beta}$ for $\alpha \neq \beta$ and each module Ae_α is isomorphic to one of the modules $Ae_{j_1}, \ldots, Ae_{j_a}$.

Theorem 2 *Let $A^b = e_A A e_A$ be a basic algebra associated to A. The algebra A^b is basic, does not depend on the choice of the sets $e_1, \ldots, e_n$ and $e_{j_1}, \ldots, e_{j_a}$, up to a K-algebra isomorphism and there is an equivalence of categories $modA \cong modA^b$.*

Definition 8 *The (Jacobson)* ***radical*** *$radA$ of A is the intersection of all the maximal right ideals in A. It is equal to the intersection of all maximal left ideals in A, and so it is a two-sided ideal. $rad^2 A$ denotes the ideal generated by all elements xy, where $x, y \in radA$.*

Lemma 1 *$a \in A$ is an element of $radA$, if and only if for all $b \in A$, $1 - ab$ and $1 - ba$ have two-sided inverses. Helpful for the computation of $radA$ is: if I is a two-sided nilpotent ideal of A, then $I \subseteq radA$.*

Definition 9 *Let A be basic and connected and $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. The* ***quiver*** *of A, denoted by Q_A, is defined as follows:*

The points of Q_A are the numbers $1, 2, ..., n$ which are in bijective correspondence with the idempotents $e_1, e_2, ..., e_n$.

Given two points $a, b \in (Q_A)_0$, the arrows $\alpha : a \to b$ are in bijective correspondence with the vectors in a basis of the K-vector space $e_b(radA/rad^2 A)e_a$.

Q_A is a finite quiver, because A is a finite dimensional algebra.

Now we can formulate the second main theorem:

Theorem 3 *Let A be basic and connected. There exists an admissible ideal I of KQ_A such that $A \cong KQ_A/I$.*