# The quiver algebra

## Basic knowledge

**Definition 1.** *Let be $\mathscr{C}$ and $\mathscr{D}$ two categories. A **covariant functor** F from $\mathscr{C}$ to $\mathscr{D}$ consists of a map $F : Ob(\mathscr{C}) \to Ob(\mathscr{D})$ and a map $F : Hom_{\mathscr{C}}(X,Y) \to Hom_{\mathscr{D}}(F(X),F(Y))$ for $X,Y \in Ob(\mathscr{C})$ such that $F(id_X) = id_{F(X)}$ for every $X \in Ob(\mathscr{C})$ and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f,g$ for which $g \circ f$ is defined.*

**Definition 2.** *Let be $F,F' : \mathscr{C} \to \mathscr{D}$ two covariant functors between two categories $\mathscr{C},\mathscr{D}$. A **natural transformation** is a map $\varphi : F \to F'$ which assignes to every $X \in Ob(\mathscr{C})$ a map $\varphi_X : F(X) \to F'(X)$ such that for every morphism $f \in Hom_{\mathscr{C}}(X,Y)$ we have:*

$$\varphi_Y \circ F(f) = F'(f) \circ \varphi_X$$

*$\varphi$ is called a **natural isomorphism** if $\varphi_X$ is an isomorphism $\forall\ X \in Ob(\mathscr{C})$.*

**Definition 3.** *Two categories $\mathscr{C}$ and $\mathscr{D}$ are **equivalent** if there exists two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that $F \circ G$ and $G \circ F$ are natural isomorphic to the corresponding identity functors.*

**Definition 4.** *Let be k a field and S a set. The **free associative k-algebra** with 1 over S is the k-vector space $k\langle S\rangle$ spanned by the monomials in S (with multliplication given by the concatenation of such monomials). We can think of it as the polynom algebra with non-commutative variables $(X_i)_{i\in S}$.*

## The quiver algebra

Let be k an algebraically closed field and $Q = (Q_o, Q_1, s, t)$ a fixed quiver. Let denote $s(\alpha)$ (resp. $t(\alpha)$) the source (resp. the target) of $\alpha \in Q_1$ in $Q_0$.

**Definition 5.** *The **quiver algebra** is the associative algebra kQ determined by the generators, $e_i$ where $i \in Q_0$ and $\alpha \in Q_1$ with the relations*

$$e_i^2 = e_i\ \forall i \in Q_0, \quad e_{t(\alpha)}\alpha = \alpha e_{s(\alpha)} = \alpha\ \forall \alpha \in Q_1,$$

*and all other products of generators are 0.*

**Proposition 1.** *The category of representaion of any quiver Q is equivalent to the category of left kQ-modules.*

**Example 1.** *The algebra of the loop L is freely generated by $\alpha$. Furthermore, the algebra of the r-loop $L_r$ (see Figure 1) is the free algebra $k\langle X_1,....,X_r\rangle$ on the r arrows.*

**Example 2.** *The algebra of the r-arrow Kronecker quiver (see Figure 2) $K_r$ is the direct sum of $k\alpha_1 \oplus .... \oplus k\alpha_r$ with $ke_i \oplus ke_j$.*

**Example 3.** *The algebra of the star shaped quiver $S_r$ (see Figure 3) is the direct sum of the two-sided ideal $k\alpha_1 \oplus .... \oplus k\alpha_r$ with the subalgebra $ke_1 \oplus .... \oplus ke_{r+1}$.*

**Definition 6.** *A **relation** of a quiver $Q$ is a subspace of $kQ$ spanned by linear combinations of paths having a common source and a common target, and of lenght at least 2.*

**Definition 7.** *The **path algebra** of a quiver with relation $(Q,I)$ is the quotient algebra $kQ/I$.*

**Remark 1.** *Every algebra can be written as the quotient of a free algebra in a trivial way. But while this point of view does not contain any information about the algebra itself, representation theory of quivers can be used to study finite dimensional algebras in detail.*
*Indeed, one can show that any finite dimensional basic algebra[1] can be written as a quotient $A = kQ/I$, where $kQ$ is the path algebra of some quiver $Q$ and $I$ is an ideal contained in $kQ_{\geq 2}$.*


Figure 1: The 2-loop $L_2$

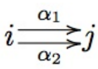


Figure 2: The 2-arrow Kronecker quiver $K_2$

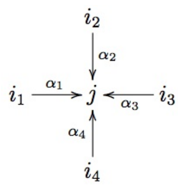


Figure 3: The quiver $S_4$

[1]Basic means, that $A/radA$ is isomorphic to a direct sum of division algebras.