Indecomposable representations, Endomorphisms and the Krull-Remak-Schmidt theorem

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In this section we consider finite dimensional representations.

Definition 1: Let $X_1, \cdots, X_r$ be a finite number of representations. A *direct sum* $X = X_1 \oplus \cdots \oplus X_r$ is a representation X together with morphisms $\iota_i : X_i \longrightarrow X$ and $\pi_i : X \longrightarrow X_i$ for $1 \leq i \leq r$, such that $\sum_{i=1}^{r} \iota_i \pi_i = id_X$ and $\pi_i \iota_i = id_{X_i}$.

Definition 2: A family of representations $X_1, ..., X_r$ of X satisfying: $X = \sum_{i=1}^{r} X_i$ and $X_i \cap \sum_{i' \neq i} X_i' = 0$ for $1 \leq i \leq r$ is called *direct sum decomposition* of X.

Lemma 1: Let $X = X_1 \oplus .... \oplus X_r$ and $Y = Y_1 \oplus .... \oplus Y_s$. Then we have induced *vector space decompositions*:

$$\bigoplus_{i=1}^{r} Hom(X_i, Y) \simeq Hom(X, Y) \simeq \bigoplus_{j=1}^{s} Hom(X, Y_j).$$

Definition 3: A representation X is called *indecomposable* if $X \neq 0$ and $X = X_1 \oplus X_2$ implies $X_1 = 0$ or $X_2 = 0$.

Definition 4: The set of morphisms $X \longrightarrow Y$ we denote by Hom(X;Y). The set of morphisms $X \longrightarrow X$ is the set of the *endomorphisms* $X \longrightarrow X$ and we write *End(X)*. Note that $(End(X), +, \circ)$ is a ring.

Lemma 2:*(Fitting)* Let X be a representation and ϕ an endomorphism:
1) For large enough r, we have $X =$Im$\phi^r \oplus$ Kerϕ^r.

2) If X is *indecomposable*, then ϕ is either an *automorphism* or *nilpotent.*

Definition 5: A ring is called *local* if the sum of two non-units is again a non-unit.

Proposition 1: A representation X is indecomposable if and only if End(X) is local. (The assumption on X to be *finite* is necessary).

Example 1: (Counterexample) Let k[t] denote the polynomial ring in one variable and consider the following representation of the Kronecker quiver. The endomorphism ring of X is isomorphic to k[t].
So the proposition 1 doesn't hold for infinite dimensional X.

Definition 6: Given a pair X,Y of representations, we define the *radical*: Rad(X,Y)= { $\phi \in$ Hom(X,Y)|$\tau\phi\sigma$ is non-invertible for every pair $\sigma : Z \longrightarrow X$ and $\tau : Y \longrightarrow Z$, with Z indecomposable}.

Lemma 3: Let X,Y be a pair of representations.
1) Rad(X,Y) is a subspace of Hom(X,Y).
2) Rad(X,$Y_1 \oplus Y_2$)=Rad(X,Y_1)$\oplus$ Rad(X,Y_2).
3) Rad($X_1 \oplus X_2, Y$)= Rad(X_1, Y)$\oplus$ Rad(X_2, Y).
4) If X and Y are indecomposable, then Hom(X,Y)$\setminus$ Rad(X,Y) equals the set of isomorphisms $X \longrightarrow Y$.

Proof: 1) Let $\phi_1, \phi_2 \in$ Rad(X,Y). Choose $\sigma \in$ Hom(Z,X) and $\tau \in$ Hom(Y,Z) with Z indecomposable. Then $\tau\phi_1\sigma$ and $\tau\phi_2\sigma$ are non-invertible, and therefore $\tau(\phi_1 + \phi_2)\sigma = \tau\phi_1\sigma + \tau\phi_2\sigma$ is non-invertible, since End(Z) is local by proposition 1. Thus $\phi_1 + \phi_2$ belongs to Rad(X,Y).
2) Let Y=$Y_1 \oplus Y_2$ and $\phi = (\phi_i) \in$ Hom(X,Y) with $\phi_i \in$ Hom(X,Y_i) for i=1,2. Choose $\phi \in$ Hom(Z,X) and $\tau = (\tau_i) \in$ Hom(Y,Z) with Z indecomposable and $\tau_i \in$ Hom(Y_i, Z) for i=1,2. Then $\tau\phi\sigma = \tau_1\phi_1\sigma + \tau_2\phi_2\sigma$.
If $\phi_i \in$ Rad(X,Y_i) for i=1,2, then $\tau_i\phi_i\sigma$ is non-invertible for i=1,2, and therefore $\tau\phi\sigma$ is non-invertible, since End(Z) is local by proposition 1. Thus ϕ belongs to Rad(X,Y). Conversely, let $\phi \in$ Rad(X,Y) and fix $i \in$ {1,2}. Then $\phi_i \in$ Rad(X,X_i) because we can put $\tau_j = 0$ for $j \neq i$ and have that $\tau_i\phi_i\sigma = \tau\phi\sigma$ is non-invertible.
3) Analogous to part 2).
4) Let $\phi \in$ Hom(X,Y)$\setminus$Rad(X,Y). Choose $\sigma \in$ Hom(Z,X) and $\tau \in$ Hom(Y,Z) with Z indecomposable such that $\tau\phi\sigma$ is invertible. Then ϕ is invertible because X is indecomposable, and τ is invertible because Y is indecomposable. Thus ϕ is invertible.

It is clear that an isomorphism $X \longrightarrow Y$ does not belong to Rad(X,Y).
□

Theorem(Krull-Remak-Schmidt): Let X be a finite dimensional representation. Then there exists a decomposition $X = X_1^{a_1} \oplus \cdots \oplus X_r^{a_r}$ with the X_i pairwise non-isomorphic indecomposable representations and each $a_i \geq 1$. If $X = Y_1^{b_1} \oplus \cdots \oplus Y_s^{b_s}$ is another such decomposition, then $r = s$ and, after reordering, $X_i \cong Y_i$ and $a_i = b_i$ for $1 \leq i \leq r$.

References

H.Krause, Representation of quivers via reflection functors.