Reflection Functors

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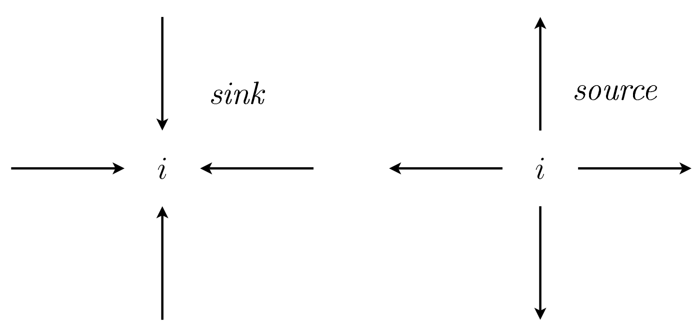
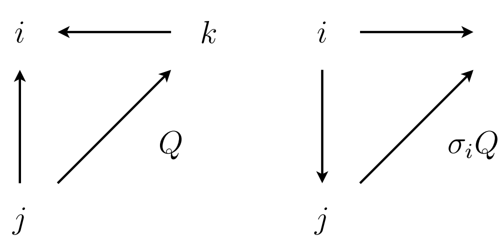
1 The Euler Form

Definition 1.1. *Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A vertex $i \in Q_0$ is called a* sink *(respectively a* source*) if there is no arrow $\alpha \in Q_1$ such that $s(\alpha) = i$ (resp. $t(\alpha) = i$).*



Given any vertex i, let $\sigma_i Q$ be the quiver obtained from Q by reversing all arrows which start or end at i.



Definition 1.2. *Let* $n = |Q_0|$. *The* Euler form *of* Q *is the bilinear form* $\langle -, - \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ *defined by*

$$\langle x, y \rangle := \langle x, y \rangle_Q = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

We then define a symmetric bilinear form $(-, -)$ *on* $\mathbb{Z}^n$ *by setting*

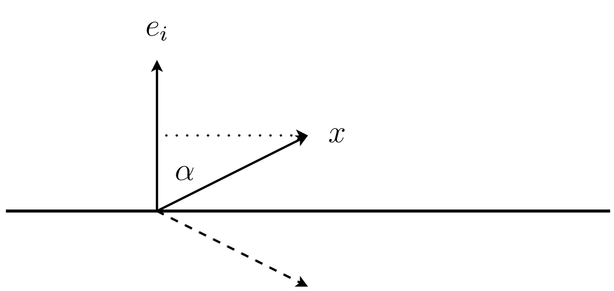
$$(x, y) := \langle x, y \rangle + \langle y, x \rangle .$$

Note that the symmetric bilinear form $(-, -)$ does not depend on the direction of the arrows of Q.

Let us now assume that Q has no loops. The *reflection with respect to a vertex* i is the linear map

$$\sigma_i : \mathbb{Z}^n \to \mathbb{Z}^n \text{ defined by } \sigma_i(x) := x - 2\frac{(x, e_i)}{(e_i, e_i)} e_i$$

where e_i is the i-th vector of the canonical basis of $\mathbb{Z}^n$. This map is well defined, since $(e_i, e_i) \neq 0$ for all $i \in Q_0$ because Q has no loops. The fact that the map σ_i is regarded as a reflection can be justified by a geometric interpretation in $\mathbb{R}^n$ (see picture). Finally, it is easy to see that σ_i is an automorphism of $\mathbb{Z}^n$ and that $\sigma_i \circ \sigma_i = \mathrm{Id}_{\mathbb{Z}^n}$.



2 The Reflection Functors S_i^+ and S_i^-

Let Q be a quiver, and let i be a vertex of Q. We now define two functors S_i^+ and S_i^-. For this purpose, fix two representations X and X' of Q, as well as a morphism $\varphi : X \to X'$.

1. If i is a sink of Q we construct

$$S_i^+ : \mathrm{Rep}(Q, \mathbb{K}) \to \mathrm{Rep}(\sigma_i Q, \mathbb{K})$$

as follows. Let $Y := S_i^+(X)$, where $Y = ((Y_i),(Y_\alpha))_{i\in Q_0,\alpha\in Q_1}$ is the following representation of $\sigma_i Q$. The vector spaces $(Y_i)_{i\in Q_0}$ are given by $Y_j = X_j$ for vertices $i \neq j$, and

$$Y_i = \operatorname{Ker}\left(\xi : \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \longrightarrow X_i\right)$$

where $\xi = (X_\alpha)$ is the map induced by the X_α with $t(\alpha) = i$.

We now define the linear maps Y_α. For an arrow α in Q with $t(\alpha) \neq i$, set $Y_\alpha := X_\alpha$. If $t(\alpha) = i$, define $Y_\alpha : Y_i \to Y_{s(\alpha)} = X_{s(\alpha)}$ as the composition

$$Y_i \overset{\check{\xi}}{\hookrightarrow} \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \xrightarrow{pr} X_{s(\alpha)}$$

where pr denotes the canonical projection onto $X_{s(\alpha)}$.

It remains to define the image of the morphism $\varphi : X \to X'$ under the functor S_i^+. Let $\psi := S_i^+(\varphi)$, where $\psi : Y \to Y'$ is the following morphism. Set $\psi_j = \varphi_j$ if $j \neq i$, and define $\psi_i : Y_i \to Y_i'$ as the restriction of the map

$$(\varphi_{s(\alpha)}) : \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \longrightarrow \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X'_{s(\alpha)}.$$

2. If i is a source of Q, we construct

$$S_i^- : \operatorname{Rep}(Q,\mathbb{K}) \to \operatorname{Rep}(\sigma_i Q,\mathbb{K})$$

in a dual fashion as follows. Let $Y := S_i^-(X)$, where we define $Y = ((Y_i),(Y_\alpha))_{i\in Q_0,\alpha\in Q_1}$ as the following representation of $\sigma_i Q$. The vector spaces $(Y_i)_{i\in Q_0}$ are given by $Y_j = X_j$, for vertices $i \neq j$, and

$$Y_i = \operatorname{Coker}\left(\xi : X_i \longrightarrow \bigoplus_{\substack{\alpha\in Q_1\\ s(\alpha)=i}} X_{t(\alpha)}\right)$$

where $\xi = (X_\alpha)$ is the map induced by the X_α with $s(\alpha) = i$.

We now define the linear maps Y_α. For an arrow α in Q with $s(\alpha) \neq i$, set $Y_\alpha := X_\alpha$. If $s(\alpha) = i$, define $Y_\alpha : Y_{t(\alpha)} = X_{t(\alpha)} \to Y_i$ as the composition

$$X_{t(\alpha)} \overset{i}{\hookrightarrow} \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X_{t(\alpha)} \overset{\check{\xi}}{\longrightarrow} Y_i$$

where i denotes the inclusion map, and $\check{\xi}$ is the projection onto $Y_i = \mathrm{Coker}(\xi)$.

The image $\psi := S_i^-(\varphi)$ of the morphism φ is defined as follows. Let $\psi_j = \varphi_j$ if $j \neq i$, and define $\psi_i : Y_i \to Y_i'$ as the map induced by

$$(\varphi_{t(\alpha)}) : \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X_{t(\alpha)} \longrightarrow \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X'_{t(\alpha)}.$$

Example 1. We now apply S_2^+ to the following quiver

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet} \qquad Q$$

with the representation given by

$$X_j = \begin{cases} \mathbb{K} & \text{if } j = 1 \\ 0 & else \end{cases}$$

and $X_\alpha = 0$ for any arrow α, i.e. all linear maps are zero.

$S_2^+ : Rep(Q, \mathbb{K}) \to Rep(\sigma_2 Q, \mathbb{K})$. Then we obtain

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \qquad \sigma_2 Q$$

- $S_2^+(X_1) = \mathbb{K} = Y_1$
- $S_2^+(X_2) = \mathrm{Ker}\xi$, with $\xi : \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X_{s(\alpha)} = \mathbb{K} \bigoplus 0 \to 0 = X_2$ and therefore $\mathrm{Ker}\xi = \mathbb{K} = Y_2$
- $S_2^+(X_3) = 0 = Y_3$

Let i be a sink of Q. Then we define a natural monomorphism

$$\iota_i X : S_i^- S_i^+ X \to X$$

by letting $(\iota_i X)_j = id_{X_j}$ for a vertex $j \neq i$, and letting $(\iota_i X)_i$ be the canonical map

$$(S_i^- S_i^+ X)_i = \mathrm{Coker}\check{\xi} \cong \mathrm{Im}\xi \to X_i.$$

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longleftarrow \overset{3}{\bullet} \quad Q$$

Example 2. Let Q be the following quiver, where $X_1 = 0$, $X_2 = \mathbb{K}^2$ and $X_3 = \mathbb{K}$. The linear map $X_1 \to X_2$ is again the zero map, while the linear map $X_3 \to X_2$ is given by the inclusion. We now look at $i_2 X$. We first apply S_2^+ to the quiver, obtaining

$$\overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \quad \sigma_2 Q$$

- $S_2^+ X_1 = X_1$
- $S_2^+ X_2 = \text{Ker}\xi$ with $\xi : \mathbb{K} \bigoplus 0 \to \mathbb{K}^2$. So we have $S_2^+ X_2 = 0$
- $S_2^+ X_3 = X_3$

We now apply the functor S_2^- to the resulting quiver and we obtain

- $S_2^- S_2^+ X_1 = S_2^- X_1 = X_1$
- $S_2^- S_2^+ X_2 = \text{Coker}\xi$ with $\xi : 0 \to \mathbb{K} \bigoplus 0$. So we have $S_2^- S_2^+ X_2 = \mathbb{K}$
- $S_2^- S_2^+ X_3 = S_2^- X_3 = X_3$

Let i be a source of Q. Then we define a natural epimorphism

$$\pi_i X : X \to S_i^+ S_i^- X$$

by letting $(\pi_i X)_j = id_{X_j}$ for a vertex $j \neq i$, and letting $(\pi_i X)_i$ be the canonical map

$$X_i \to \text{Im}\xi \cong \text{Ker}\hat{\xi} = (S_i^+ S_i^- X)_i.$$

Lemma 2.1. *S_i^+ and S_i^- are functors, that is, $S_i^\pm id_X = id_{S_i^\pm}$ for every representation X and $S_i^\pm(\phi\psi) = (S_i^\pm \phi)(S_i^\pm \psi)$ for every pair $\phi : X \to Y$ and $\psi : Y \to Z$ of morphisms.*

Proof. Clear ☐

Lemma 2.2. *Let X, X' be representation of Q and i be a vertex.*
(1) $S_i^\pm(X \bigoplus X') = S_i^\pm X \bigoplus S_i^\pm X'$
(2) $X = (S_i^- S_i^+ X) \bigoplus \text{Coker}\iota_i X$ *and* $X = (S_i^+ S_i^- X) \bigoplus \text{Ker}\pi_i X$
(3) If Coker $\iota_i X = 0$, *then* dim $S_i^+ X = \sigma_i(\text{dim} X)$.
(4) If Ker $\pi_i X = 0$, *then dim* $S_i^- X = \sigma_i(dim X)$.

Proof. Next time. □

Example 3. Let us now apply the functor S_2^- to the quiver S_2^+Q of example 1, then we have:

- $S_2^-(Y_1) = \mathbb{K} = X_1$
- $S_2^-(Y_2) = \text{Coker}\xi$, with $\xi : Y_i = \mathbb{K} \to \mathbb{K} \bigoplus 0 = \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} Y_{t(\alpha)}$ and for this $\text{Coker}\xi = 0 = X_3$
- $S_2^-(Y_3) = 0 = X_3$

Since $\text{Coker}\xi = 0$, we can apply point (3) of Lemma 2.2 to check the result

$$\begin{aligned}
\sigma_2(\text{dim}X) &= \text{dim}X - \frac{2(\text{dim}(X), e_2)}{(e_2, e_2)} e_2 \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2(\langle \text{dim}X, e_2 \rangle + \langle e_2, \text{dim}X \rangle)}{2 \langle e_2, e_2 \rangle} e_2 \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{-1+0}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \text{dim}S_2^+X
\end{aligned}$$