Quiver Representations and Gabriel's Theorem

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Abstract

In this seminar presentation, we will introduce the categorical aspect of quiver representation and introduce the indecomposable representations. The latter play an important role in Gabriel's Theorem, which allows to determine the quivers with finitely many indecomposable representations. Next week we will connect indecomposable representations with the initial cluster variables by a bijective map - a reason to start studying indecomposable representations.

1 Quiver Representation

Definition 1 *Let k be an algebraically closed field and $Q = (Q_0, Q_1)$ be a finite quiver without oriented cycles. A representation V of Q is a set of*

- *finite dimensional k-vector spaces V_i for each vertex i of Q*
- *linear maps $V_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$ of Q*

Example:

Let Q be the following quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & & 3 \end{tikzcd}$$

A representation of Q is

$$\begin{tikzcd} & V_2 \arrow[dr, "V_\beta"] & \\ V_1 \arrow[ur, "V_\alpha"] \arrow[rr, "V_\gamma"] & & V_3 \end{tikzcd}$$

Definition 2 *A morphism of a representation $f : V \to W$ (V, W representations of Q) is a collection of k-linear maps $f_i : V_i \to W_i$ for each vertex i of Q such that the square*

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "f_i"] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_\alpha"] & W_j \end{tikzcd}$$

commutes for all arrows $\alpha : i \to j$ of Q.

The composition of morphisms is defined in the natural way. Then, we obtain the category of representations $rep(Q)$.

Lemma 1 *A morphism $f : V \to W$ of this category is an isomorphism if and only if its components f_i are invertible for all vertices i of Q_0.*

<u>Example:</u>

Let

$$V : V_1 \xrightarrow{V_\alpha} V_2$$

be a representation of Q. By choosing basis in the spaces V_1 and V_2, we find an isomorphism of representations.

$$\begin{tikzcd} V_1 \arrow[r, "V_\alpha"] \arrow[d] & V_2 \arrow[d] \\ k^n \arrow[r, "A"] & k^p \end{tikzcd}$$

By abuse of notation, we denote A by the multiplication by a $p \times n$ matrix A. We know that we have

$$PAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} =: I_r \oplus 0$$

The matrices P and Q denote invertible matrices and r is the rank of A.

Then, we get

$$\begin{tikzcd} k^n \arrow[r, "A"] \arrow[d, "Q^{-1}"] & k^p \arrow[d, "P"] \\ k^n \arrow[r, "1_r \oplus 0"] & k^p \end{tikzcd}$$

Lemma 2 *Let Q be a finite quiver. Then $rep_k(Q)$ is an abelian category.*

Proof:
See lecture or [IA].
For more information about categories, please read Prof. Baur's lecture notes [KB]

1.0.1 Remark:

The direct sum $V \oplus W$ is then the representation given by $(V \oplus W)_i = V_i \oplus W_i$ (componentwise) and $(V \oplus W)_\alpha = V_\alpha \oplus W_\alpha$ for vertices i and arrows α of Q.

Example:

The above representation in normal form $I_r \oplus 0$ is isomorphic to the direct sum

$$(k \xrightarrow{1} k)^r \oplus (k \xrightarrow{1} 0)^{n-r} \oplus (0 \xrightarrow{1} k)^{p-r}$$

Definition 3 *A subrepresentation V' of a representation V which is given by a family of subspaces $V_i' \subset V_i$, $i \in Q_0$, such that the image of V_i' under V_α is contained in V_j' for each arrow $\alpha : i \to j$ of Q.*

Definition 4 *A sequence*

$$0 \to U \to V \to W \to 0$$

of representations and morphisms is a short exact sequence if the sequence

$$0 \to U_i \to V_i \to W_i \to 0$$

is exact for each vertex i in Q. A sequence of representations and morphisms between them is exact, if the image of one morphism is equal to the kernel of the next.

Definition 5 *A representation V is simple if it is non-zero and if for each subrepresentation V' of V we have $V' = 0$ or $V/V' = 0$. Equivalently a representation is simple if it has exactly two subrepresentations.*

Definition 6 *A representation V is indecomposable if it is non zero and in each decomposition $V = V' \oplus V''$ we have $V' = 0$ or $V'' = 0$. Equivalently a representation is indecomposable if it has exactly two direct factors.*

1.0.2 Three Examples:

- The representations $k \to 0$ and $0 \to k$ are simple.
- The representation $V = (k \xrightarrow{1} k)$ is not simple, because it has the non trivial subrepresentation $0 \to k$
- But $V = (k \xrightarrow{1} k)$ is indecomposable. Since each endomorphism $f: V \to V$ is given by two equal components $f_1 = f_2$ such that the endomorphism algebra of V is one-dimensional. If V was a direct sum $V_1 \oplus V_2$ for two non-zero subspaces, the endomorphism algebra of V would contain the product of the endomorphism algebras V_1 and V_2 and thus would have at least dimension 2.

Definition 7 *A ring is called local, if the non-invertible elements form an ideal.*

Unique Decomposition Theorem: 1 *Let V be a representation.*

- *The representation V is indecomposable if and only if the endomorphism algebra $End(V)$ is local.*
- *Each representation decomposes into a finite sum of indecomposable representations. This decomposition is unique up to isomorphisms and permutations.*

For the proof, see Krause's Script [KR] on pages 6 and 7.

2 Gabriel's Theorem

As mentioned in the abstract, Gabriel's theorem allows a characterization of quivers with finitely many isomorphism classes of representations. First, a few definitions are needed.

Definition 8 *For any representation V, we can define the dimension vector*

$$v = (dim(V_1), dim(V_2), \ldots, dim(V_n))$$

For example, the dimension vector of a representation

$$k^p \to k^r \to k^s$$

is simply $v = (p, r, s)$.

Definition 9 *The Tits form is a quadratic form defined on the dimension vector by*

$$q_Q(v) = \sum_{i \in Q_0} v_i^2 - \sum_{\alpha \in Q_1} v_{s(\alpha)} v_{t(\alpha)}$$

Obviously, the Tits form is independent of the orientation of the arrows.

Definition 10 *A quiver Q is representation-finite, if there are only finitely many indecomposable representations up to isomorphism.*

Definition 11 *A vector $v \in \mathbb{Z}^{Q_\alpha}$ is called a root, if it satisfies*

$$q_Q(v) = 1$$

Definition 12 *A root $v = (v_1, \ldots, v_n)$ is positive, if $v_i \geq 0$ for $i \in \{1, \ldots, n\}$.*

Gabriel's Theorem: 1 *Let Q be a connected quiver and k be an algebraically closed field. Equivalent are:*

1. *Q is representation-finite*
2. *q_Q is positive definite*
3. *The underlying graph is a simply laced Dynkin diagram Δ.*

Moreover, the isomorphism classes of indecomposable representations can naturally be mapped to the set of positive roots of the Tits form q_Q.

The proof of $iii \to i$ and $i \to iii$ is long and requires reflection functors. If interested, they can be found on page 290-292 of [IA]. The proofs of $i \to ii$ and $ii \to iii$ are in [KW].

References

[KB] **Karin Baur:**
Skript: Homologische Algebra und modulare Darstellungstheorie
http://www.math.ethz.ch/~baur/Teaching/SkriptHS08/HA-Skript.pdf

[IA] **Ibrahim Assem, Daniel Simson, and Andrzej Skowronski:**
Elements of the representation theory of associative algebras Vol. 1
London Mathematical Society Student Texts, vol.65 Cambridge University Press, Cambridge 2006

[BK] **Bernhard Keller:**
Cluster algebras, quiver representations and triangulated categories, May 2009
http://front.math.ucdavis.edu/0807.1960

[KW] **Kirstin Webster:**
Quiver Representation and Gabriel's Theorem
http://www.math.neu.edu/~king_chris/webster.pdf

[KR] **Henning Krause:**
Representations of Quivers via Reflection Functors
http://arxiv.org/PS_cache/arxiv/pdf/0804/0804.1428v1.pdf