HODGE THEORY, CHAPTER 11

Chern classes and Euler classes. Let E be a real oriented vector bundle of rank r on a differentiable manifold X. We define its Euler class $e(E)$ in the following way: the section 0 of E provides a submanifold

$$X \overset{0}{\hookrightarrow} E$$

of E. This submanifold has an oriented normal bundle and hence possesses a class $[X] \in H^r(E, \mathbb{Z})$. Then we define

$$e(E) := 0^*[X].$$

(a) Let L be a holomorphic line bundle on a complex manifold X. We can see L as a real differentiable vector bundle $L_{\mathbb{R}}$ of rank 2. Show that $c_1(L) = e(L_{\mathbb{R}})$ in $H^2(X, \mathbb{Z})$. (Use the Lelong formula on L.)

(b) Let E be an oriented real differentiable vector bundle of rank $2k$ which is a direct sum of k complex line bundles L_i, more precisely of their underlying real vector bundles $L_{i,\mathbb{R}}$. Show that $e(E) = e(L_{1,\mathbb{R}}) \cup \cdots \cup e(L_{k,\mathbb{R}})$.

(c) Deduce from (a), (b) and the splitting principle that for any holomorphic vector E of rank k on X, with underlying real vector bundle $E_{\mathbb{R}}$ of rank $2k$, we have

$$c_k(E) = e(E_{\mathbb{R}}) \text{ in } H^{2k}(X, \mathbb{Z}).$$

(Note that this remains true in the differentiable case.)

(d) Let $Z \subset X$ be a complex submanifold of codimension r of a complex manifold X. Show with the help of the construction of $[Z]$ in a tubular neighbourhood of Z in X and of (c) that

$$[Z]_{|Z} = c_r(N_{Z/X}) \text{ in } H^{2r}(Z, \mathbb{Z}).$$

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