

2)(a) $p: \mathbb{C} \times \mathcal{H} \rightarrow \mathbb{C} \times \mathcal{H} / \mathbb{Z}^2$
 $(z, \tau) \mapsto [z/\tau]$

$U := \{(x + \tau y, \tau) \mid x, y \in (0, 1), \tau \in \mathcal{H}\}$

plus injective: $[x + \tau y, \tau] = [x' + \tau' y', \tau']$

$\Rightarrow \tau = \tau' \wedge x + m + \tau(y + n) = x' + \tau'y'$

x, y, τ linearly indep. over \mathbb{R}

$\Rightarrow x + m = x' \wedge y + n = y'$

$x, y, y' \in (0, 1) \Rightarrow (m, n) = 0$

$\Rightarrow p|_U: U \rightarrow p(U)$ bijective

Furthermore: p opens (V open in $\mathbb{C} \times \mathcal{H} \Rightarrow p^{-1}(p(V))$)

$= \bigcup_{(m,n) \in \mathbb{Z}^2} V$ open in $\mathbb{C} \times \mathcal{H}$, i.e.

$p(V)$ open

In the same way we get homeomorphisms

by restricting p to

$\cup (z, 0) + U, (z, 0) + U, (z + \frac{\pi}{2}, 0) + U$
 $\quad \quad \quad \text{or const.!}$

p continuous (by def. of the quotient topology)

coordinate charts: $(p|_V)^{-1}: p(V) \rightarrow V \subset \mathbb{C}^2$

change of coordinates: e.g. $(p|_{U'})^{-1} \circ ((p|_{(z, 0) + U})^{-1})^{-1}((x + \tau y, \tau))$
 $= (p|_U)^{-1}([z/\tau]) = \begin{cases} (z/\tau) & x < 1 \\ (z - 1/\tau) & x > 1 \end{cases}$

in general $(p|_W)^{-1} \circ p|_V(z/\tau) = (z + \underbrace{m(z) + n(z)\tau}_{\text{locally constant}}, \tau)$

(b) ϕ is well-defined and continuous, since it is induced by $\mathbb{C} \times \mathcal{H} \xrightarrow{\text{pr}_2} \mathcal{H}$.

ϕ holomorphic: e.g. $\phi \circ (p|_U)^{-1} : U \rightarrow \mathcal{H}$ this local description also shows that its holomorphic differential is surjective.

$X_\alpha \cong \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$

\mathbb{Z}^2 -equivariant

if \mathbb{Z}^2 acts on \mathbb{C} by $(m, n) \cdot z = z + m + n\tau$

$\mathbb{C} \rightarrow \mathbb{C} \times \mathcal{H}$

\downarrow

$\mathbb{C}/\mathbb{Z} + \tau \mathbb{Z} \rightarrow X$

$[z] \mapsto [z/\tau]$

has image X_α , an inverse for $\mathbb{C}/\mathbb{Z} + \tau \mathbb{Z} \rightarrow X_\alpha$ is $[z/\tau] \mapsto [z]$

in charts, both maps are holomorphic because the coordinate charts are local sections of the respective canonical projections.

$\phi^{-1}(K) = \{[z/\tau] \mid z = x + \tau y, x, y \in [0, 1], \tau \in K\}$
compact for $K \subseteq \mathcal{H}$ compact

(c) $\mathbb{C} \times \mathcal{H} \xrightarrow{T} \mathbb{C} \times \mathcal{H}$

\mathbb{Z}^2 acts by $(m, n) \cdot (z, \tau) = (z + m + n\tau, \tau)$

\mathbb{Z}^2 acts by $X \xrightarrow{T} X \times \mathcal{H}$
 $(m, n) \cdot (x, \tau) \mapsto (x + m + n\tau, \tau)$

Ansatz: $T = (T_0, T_1)$, T_0 R-linear

We want T smooth

T \mathbb{Z}^2 -equivariant

$\text{pr}_2 \circ T = \text{pr}_2$ (by def. of ϕ and \mathbb{Z}^2 -actions, which are trivial in the τ -component)

$$\Rightarrow \tilde{T}(x+iy, \tau) = (x\tilde{T}_0(1, \tau) + y\tilde{T}_0(i, \tau), \underbrace{\tilde{T}_0(x+iy, \tau)}_{=\tau})$$

$$\tilde{T} \text{ is } \mathbb{Z}^2\text{-equivariant} \Leftrightarrow \tilde{T}_0(z+1, \tau) = \tilde{T}_0(z, \tau) + 1$$

$$\tilde{T}_0(z+\tau_1, \tau) = \tilde{T}_0(z, \tau) + i \quad (\text{for all } (z, \tau))$$

$$\Leftrightarrow (x+1)\tilde{T}_0(1, \tau) + y\tilde{T}_0(i, \tau) = x\tilde{T}_0(1, \tau) + y\tilde{T}_0(i, \tau) + 1$$

$$(x+\tau_1)\tilde{T}_0(1, \tau) + (y+\tau_2)\tilde{T}_0(i, \tau) = x\tilde{T}_0(1, \tau) + y\tilde{T}_0(i, \tau) + i$$

writing $\tau = \tau_1 + i\tau_2$

$$\Leftrightarrow \tilde{T}_0(1, \tau) = 1 \quad \tau_1 + \tau_2 \tilde{T}_0(i, \tau) = i$$

$$\Leftrightarrow \tilde{T}(x+iy, \tau) = (x + \frac{\tau_2 - i}{\tau_2} y, \tau)$$

$$\tilde{T}^{-1}(w, \tau) \stackrel{w=u+iv}{=} (u + \tau v, \tau)$$

$$(d) \quad \tilde{\sigma}|_{X_\tau}(x+iy, \tau) = x + \frac{i-\tau_1}{\tau_2} y, \quad \tilde{\sigma}|_{X_\tau}^{-1}(w, \tau) = u + \tau v \stackrel{C \cong \mathbb{R}^2}{\longleftrightarrow} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

↑ coord.
on X_τ

$$(\tilde{\sigma}|_{X_\tau}^{-1})^* dx = d(u + \tau_1 v) = du + \tau_1 dv$$

$$(\tilde{\sigma}|_{X_\tau}^{-1})^* dy = d(\tau_2 v) = \tau_2 dv$$

$$F^1 H^1(X_\tau) = H^{1,0}(X_\tau) = \mathbb{C} \cdot \underline{dz}$$

can be defined using $X_\tau \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and dz on $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$
which is defined (locally) to be $(p|_V)^* dz$

$$\mathcal{P}^{1,1}(\tau) = (\tilde{\sigma}|_{X_\tau}^{-1})^* \mathbb{C} \cdot \underline{dz} = \mathbb{C} \cdot \left(\underline{du} + \tau \underline{dv} \right) = \mathbb{C} \cdot \left(\frac{1-i\tau}{2} \underline{dw} + \frac{1+i\tau}{2} \underline{d\bar{w}} \right)$$

$= dx + dy \quad = \frac{dw + d\bar{w}}{2} \quad = \frac{dw - d\bar{w}}{2i}$

$= 0$ for $\tau = 0$, thus never zero for $\tau \in \mathbb{R}$

$C^\infty(P^1(H^1(X_\tau, \mathbb{C})), \mathbb{R})$

$$\frac{d}{d\tau} \Big|_{\tau=\tau_0} \mathcal{P}^{1,1}(\tau) = \frac{d}{d\tau} \Big|_{\tau=\tau_0} \varphi(\mathcal{P}^{1,1}(\tau)) \cdot \frac{\partial}{\partial b} \Big|_{\mathcal{P}^{1,1}(\tau)}$$

chart φ for $P^1(H^1(X_\tau, \mathbb{C}))$:
 $= \mathbb{C} dw \oplus \mathbb{C} d\bar{w}$

$$\varphi: \{ C(\alpha dw + \beta d\bar{w}) \mid \alpha \neq 0 \} \rightarrow \mathbb{C}$$

$$\mathbb{C}(dw + \beta d\bar{w}) \mapsto \beta$$

$$\left(\frac{d}{dt} \Big|_{t=t_0} \mathcal{P}^{1,1}(t) \right) f = \frac{d}{dt} \Big|_{t=t_0} f \circ \varphi^{-1} \circ \varphi \circ \mathcal{P}^{1,1}(t)$$

$$= \frac{\partial(f \circ \varphi^{-1})}{\partial b} (\varphi \circ \mathcal{P}^{1,1}(t_0)) \cdot \frac{d(\varphi \circ \mathcal{P}^{1,1}(t))}{dt} \Big|_{t=t_0}$$

and $\frac{d}{dt} = \frac{1}{2} \left(\frac{d}{d\tau_1} - i \frac{d}{d\tau_2} \right)$

$$= \frac{d}{d\tau} \Big|_{\tau=\tau_0} \left(\frac{1+i\tau}{2} \right) \cdot \frac{\partial}{\partial b} \Big|_{\mathcal{P}^{1,1}(\tau_0)}$$

$$= \frac{\partial(1-i\tau) - (1+i\tau) \cdot i}{(1-i\tau)^2} \Big|_{\tau=\tau_0} \cdot \frac{\partial}{\partial b} \Big|_{\mathcal{P}^{1,1}(\tau_0)} = \frac{2i}{(1-i\tau_0)^2} \frac{\partial}{\partial b} \Big|_{\mathcal{P}^{1,1}(\tau_0)}$$

Lemma: W complex vectorspace, S the tautological bundle of $\mathbb{P}^1(W)$,
 $E = (\mathbb{P}^1(W) \times W)/S$.

$$T\mathbb{P}^1(W) \xrightarrow{\cong} \text{Hom}_S(S, E) \quad \text{via} \quad T_L\mathbb{P}^1(W) \rightarrow \text{Hom}_L(L, W/L)$$

(this is Lemma 4.22 of P. Griffiths' Periods of Integrals on Algebraic Manifolds I, Amer. Math. J. (1968) 568-626)

$$\begin{aligned} \theta = \frac{d}{dz}|_{z=z_0} L_z &\mapsto (l \mapsto \frac{d}{dz}|_{z=z_0} \bar{z} z + L) \\ \text{where } L_z &\in \text{holomorphic curves in } \mathbb{P}^1(W) \\ L_0 &= L \end{aligned}$$

where $\bar{z} z \in L_z, \bar{z} = l$

\bar{z} holomorphic curve in W

$$\left. \frac{d}{dz} \right|_{z=z_0} C(dw + \frac{1+iz_0}{1-iz_0} d\bar{w}) \leftrightarrow \left(L = \mathcal{I}^{1,1}(z_0) = C(dw + \frac{1+iz_0}{1-iz_0} d\bar{w}) \rightarrow H^1(X_i, \mathbb{C})/L \right)$$

$$\lambda(dw + \frac{1+iz_0}{1-iz_0} d\bar{w}) \mapsto \left. \frac{d}{dz} \right|_{z=z_0} \lambda(dw + \frac{1+iz_0}{1-iz_0} d\bar{w}) + L$$

(the goal here is to see how the identification of the Lemma works in a very simple example)

$$\begin{aligned} \tilde{\lambda}(dw + z_0 dv) &= \tilde{\lambda}\left(\frac{1-iz_0}{2} dw + \frac{1+iz_0}{2} d\bar{w}\right) = \tilde{\lambda}\left(\frac{1-iz_0}{2}\right)(dw + \frac{1+iz_0}{1-iz_0} d\bar{w}) \\ &\mapsto \tilde{\lambda}\left(\frac{1-iz_0}{2}\right) \left. \frac{d}{dz} \right|_{z=z_0} (dw + \frac{1+iz_0}{1-iz_0} d\bar{w}) + L \\ &= \tilde{\lambda}\left(\frac{1-iz_0}{2}\right) \frac{2i}{(1-iz_0)^2} d\bar{w} + L \\ &= \tilde{\lambda} \frac{\partial}{\partial z_0} d\bar{w} + L \end{aligned}$$

$$\mathcal{D}^{1,1}\left(\left. \frac{\partial}{\partial z} \right|_{z_0}\right)(du + z_0 dv) = \left. \frac{d}{dz} \right|_{z=z_0} \bar{z}_0 + C \cdot (du + z_0 dv)$$

\bar{z}_0 curve in $H^1(X_0, \mathbb{C})$
with $\bar{z}_{z_0} = du + z_0 dv$,
 $\bar{z}_0 \in \mathcal{I}^{1,1}(z) = C \cdot (du + z_0 dv)$

$$\begin{aligned} &= \left. \frac{d}{dz} \right|_{z=z_0} (du + z_0 dv) + C \cdot (du + z_0 dv) \\ &= dv + C \cdot (du + z_0 dv) \end{aligned}$$

$$\frac{i}{1-iz_0} dw - dv \in L$$

$$= \frac{i}{1-iz_0} dw - \frac{dv}{1-iz_0} = \frac{dw - d\bar{w}}{2i} = -\frac{1}{2i} \left(-2i \cdot \frac{\partial}{\partial z_0} d\bar{w} + dw - d\bar{w} \right)$$

$$= \left(\frac{z_0}{1-iz_0} - \frac{1-iz_0}{1-iz_0} \right) d\bar{w}$$

$$= \frac{2-(1-iz_0)}{1-iz_0} d\bar{w}$$

$$= \frac{1+iz_0}{1-iz_0} d\bar{w}$$