Integral Representation Formula, Boundary Integral Operators and Calderón projection

Seminar BEM on Wave Scattering

Franziska Weber

ETH Zürich

October 22, 2010

Outline

- Integral Representation Formula
- Newton Potential
- Single Layer Potential and Boundary Integral Operators associated to it
- Double Layer Potential and Boundary Integral Operators associated to it
- Calderón projector

Casting

Potentials		Boundary Integral Operators
Newton Potential $\mathcal{N}^0$		
		V_0
Single Layer Potential Ψ^0_{SL}	$\Longleftrightarrow$	W_0
		K_0'
Double Layer Potential Ψ^0_{DL}		K_0

Fundamental Solution of the Laplace Equation

Consider the Poisson Equation for $d = 2, 3$

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^d, \tag{1}$$

where we assume Ω to be a Lipschitz domain. The fundamental solution of the Laplace operator is given by

$$G_0(x,y) = \begin{cases} -\frac{1}{2\pi} \log|x-y| & \text{for } d = 2, \\ \frac{1}{4\pi} \frac{1}{|x-y|} & \text{for } d = 3. \end{cases} \tag{2}$$

$\rightarrow$ How can this be derived?

Fundamental Solution of the Laplace Equation

Look for radial solution of homogeneous equation

$$-\Delta u(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^d. \tag{3}$$

i.e. we assume the solution is of the form $u(x) = v(|x|) = v(r)$. Then for $r > 0$

$$\frac{\partial}{\partial x_i} r = \frac{x_i}{r} \quad \text{and} \quad \frac{\partial^2}{\partial x_i^2} r = \frac{1}{r} - \frac{x_i^2}{r^3}$$

which yields

$$\frac{\partial}{\partial x_i} v(r) = v'(r)\frac{x_i}{r} \quad \text{and}$$
$$\frac{\partial^2}{\partial x_i^2} v(r) = v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right) + v''(r)\frac{x_i^2}{r^2}.$$

Therefore (3) is equivalent to (assuming $u(x) = v(|x|) = v(r)$)

$$v''(r) + \frac{(d-1)}{r} v'(r) = 0. \tag{4}$$

Fundamental Solution of the Laplace Equation

Multiply (4) by r^{d-1} we get

$$0 = r^{d-1}v''(r) + r^{d-2}(d-1)v'(r) = (r^{d-1}v'(r))'. \tag{5}$$

Integrate twice to obtain (for $r > 0$)

$$v'(r) = \frac{c}{r^{d-1}}$$

$$v(r) = \begin{cases} a \log r + b & \text{for } d = 2, \\ \frac{c}{r^{d-2}} & \text{for } d \geq 3. \end{cases} \tag{6}$$

Choosing the constants appropriately, we obtain (2). Notice that $u(x) = v(|x|)$ is smooth away from zero and we can write $\Delta u(x) = \delta_0(x)$.

Integral representation formula

- Recall Green's second identity

$$\int_\Omega (u\Delta v - \Delta u\, v)dy = \int_\Gamma (\gamma_D u \gamma_N v - \gamma_N u \gamma_D v)dS, \tag{7}$$

  where $\Gamma = \partial\Omega$ and γ_D, γ_N denote the interior Dirichlet and the Neumann traces respectively.

- Insert $G_0(x,y)$ for $v(y)$, $-\Delta u(x) = f(x)$ (Poisson's equation) and use that $\Delta G_0(x,y) = -\delta_0(x-y)$ in the sense of distributions to obtain

$$-\int_\Omega u(y)\delta_0(x-y)\,dy - \int_\Omega \Delta u(y) G_0(x,y)\,dy =$$
$$\int_\Gamma (\gamma_D u(y)\gamma_{N,y} G_0(x,y) - \gamma_N u(y)\gamma_D G_0(x,y))\,dS(y), \quad x \in \Omega. \tag{8}$$

Integral representation formula

- So,

$$u(x) = \int_\Gamma (\gamma_N u(y) \gamma_D G_0(x,y) - \gamma_D u(y) \gamma_{N,y} G_0(x,y))\, dS(y) \\ + \int_\Omega G_0(x,y) f(y)\, dy, \quad x \in \Omega, \tag{9}$$

Hence, if the integrals are all well defined and we know the Cauchy data $(\gamma_D u(x), \gamma_N u(x))$ for $x \in \Gamma$ we have found a solution to Poisson's equation (1) in Ω.

Integral representation formula

- Define for a compactly supported function f the **volume** or **Newton potential** by

$$(\mathcal{N}^0 f)(x) := \int_{\mathbb{R}^d} G_0(x,y) f(y)\, dy \quad \text{for } x \in \mathbb{R}^d \tag{10}$$

- and the **Single Layer Potential** for a function $\phi \in L^1(\Gamma)$,

$$(\Psi^0_{\mathrm{SL}} \phi)(x) := \int_\Gamma \gamma_D G_0(x,y) \phi(y)\, dS(y), \quad x \in \mathbb{R}^d \backslash \Gamma. \tag{11}$$

Integral representation formula

- For $\phi \in L^1(\Gamma)$ the **Double Layer Potential**

$$(\Psi^0_{\mathrm{DL}}\phi)(x) := \int_\Gamma \gamma_{N,y} G_0(x,y)\phi(y)\, dS(y), \quad x \in \mathbb{R}^d \backslash \Gamma. \tag{12}$$

- For generalized functions we define them by

$$\Psi^0_{\mathrm{SL}} := \mathcal{N}^0 \circ \gamma_D^* \quad \Psi^0_{\mathrm{DL}} := \mathcal{N}^0 \circ \gamma_N^* \tag{13}$$

- Rewrite equation (9) as (assuming f is compactly supported in Ω)

$$u(x) = (\mathcal{N}^0 f)(x) - \Psi^0_{\mathrm{DL}}(\gamma_D u)(x) + \Psi^0_{\mathrm{SL}}(\gamma_N u)(x) \quad \text{for } x \in \mathbb{R}^d \backslash \Gamma. \tag{14}$$

Aim: Show well-definedness, boundedness and useful properties of the operators $\mathcal{N}^0$, Ψ^0_{SL} and Ψ^0_{DL} and hence derive boundary integral equations for $u(x)$ in order to find the Cauchy data $(\gamma_D u, \gamma_N u)$.

The Newton Potential

- $G_0(x,y)$ is not integrable in $\mathbb{R}^d$ for $d=2,3$ and hence $\mathcal{N}^0 f$ might not be defined as the integral does not exist for arbitrary f (e.g. consider $f \equiv 1$ and $d=2$).
- But for $\phi, \psi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\left\langle \mathcal{N}^0 \phi, \psi \right\rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^d} \overline{\psi(x)} \int_{\mathbb{R}^d} G_0(x,y)\phi(y)dydx = \left\langle \phi, \mathcal{N}^0 \psi \right\rangle_{\mathbb{R}^d} \quad (15)$$

  using Fubini and that $G_0(x,y)$ is symmetric, real valued.
- Hence, $\mathcal{N}^0 \phi \in \mathcal{S}(\mathbb{R}^d)$ and we can define the Newton potential as an operator $\mathcal{N}^0 : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ by

$$\left\langle \mathcal{N}^0 f, \psi \right\rangle_{\mathbb{R}^d} := \left\langle f, \mathcal{N}^0 \psi \right\rangle_{\mathbb{R}^d} \quad \forall \psi \in \mathcal{S}(\mathbb{R}^d) \quad (16)$$

The Newton Potential

- We have

Theorem

The Newton Potential

$$\mathcal{N}^0 : H^s_{comp}(\mathbb{R}^d) \to H^{s+2}_{loc}(\mathbb{R}^d) \tag{17}$$

is continuous for all $s \in \mathbb{R}$.

The Newton Potential

- Defining

$$(\mathcal{N}^0 f)(x) := \int_\Omega G_0(x,y) f(y)\, dy \quad \text{for } x \in \mathbb{R}^d. \tag{18}$$

  we have

Theorem

For $s \in [-2, 0]$, the Newton Potential $\mathcal{N}^0 : \widetilde{H}^s(\Omega) \to H^{s+2}(\Omega)$ is a continuous map, i.e.

$$\|\mathcal{N}^0 f\|_{H^{s+2}(\Omega)} \leq c \|f\|_{\widetilde{H}^s(\Omega)} \quad \forall f \in \widetilde{H}^s(\Omega) \tag{19}$$

The Newton Potential

Proof.

1. Show claim for $\phi \in C_0^\infty(\Omega)$ and extend it to the completion with respect to the norm on $\widetilde{H}^s(\Omega)$ in the end.
2. Fouriertransform, cut-off functions, splitting of the integral and estimating terms separately, duality arguments

□

Theorem

$\mathcal{N}^0\widetilde{f}$ is a generalized solution of the partial differential equation

$$-\Delta_x(\mathcal{N}^0\widetilde{f})(x) = \widetilde{f}(x) = \begin{cases} f(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^d \backslash \overline{\Omega}. \end{cases} \tag{20}$$

Proof.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Use integration by parts, Fubini's theorem and the symmetry of the fundamental solution to obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} \left(-\Delta_x(\mathcal{N}^0\widetilde{f})(x)\right)\phi(x)dx &\overset{\text{IBP}}{=} \int_{\mathbb{R}^d} (\mathcal{N}^0\widetilde{f})(x)\left(-\Delta_x\phi(x)\right)dx \\
&\overset{\text{Def.}}{=} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_0(x,y)\widetilde{f}(y)dy\left(-\Delta_x\phi(x)\right)dx \\
&\overset{\text{sym.}}{=} \int_{\mathbb{R}^d} \widetilde{f}(y)\int_{\mathbb{R}^d} G_0(y,x)\left(-\Delta_x\phi(x)\right)dxdy \\
&\overset{\text{IBP}}{=} \int_{\mathbb{R}^d} \widetilde{f}(y)\int_{\mathbb{R}^d} \left(-\Delta_x G_0(y,x)\right)\phi(x)dxdy \\
&= \int_{\mathbb{R}^d} \widetilde{f}(y)\int_{\mathbb{R}^d} \delta_0(x-y)\phi(x)dxdy \\
&= \int_{\mathbb{R}^d} \widetilde{f}(y)\phi(y)dy
\end{aligned}$$

Finally use density of $C_0^\infty(\mathbb{R}^d)$ restricted to Ω in $H^s(\Omega)$. □

The Newton Potential

- Applying the interior trace operator

$$\gamma_D(\mathcal{N}^0 f)(x) = \lim_{\Omega \ni \tilde{x} \to x \in \Gamma} (\mathcal{N}^0 f)(\widetilde{x}) \tag{21}$$

defines a linear bounded operator

$$\gamma_D \mathcal{N}^0 : \widetilde{H}^{-1}(\Omega) \to H^{1/2}(\Gamma) \tag{22}$$

i.e. it satisfies

$$\|\gamma_D \mathcal{N}^0 f\|_{H^{1/2}(\Gamma)} \leq c\|f\|_{\widetilde{H}^{-1}(\Omega)} \quad \forall f \in \widetilde{H}^{-1}(\Omega) \tag{23}$$

- One can show

Lemma

For $f \in L^\infty(\Omega)$ it holds

$$\gamma_D(\mathcal{N}^0 f)(x) = \int_\Omega G_0(x,y) f(y) dy \tag{24}$$

The Newton Potential

- For the application of the interior Neumann trace, we have

Lemma

The operator $\gamma_N \mathcal{N}^0 : \widetilde{H}^{-1}(\Omega) \to H^{-1/2}(\Gamma)$ *is bounded, i.e.*

$$\|\gamma_N \mathcal{N}^0 f\|_{H^{-1/2}(\Gamma)} \leq c \|f\|_{\widetilde{H}^{-1}(\Omega)} \quad \forall f \in \widetilde{H}^{-1}(\Omega) \tag{25}$$

The proof uses the inverse trace theorem (i.e. that for a given $w \in H^{1/2}(\Gamma)$ there is a bounded extension $\mathcal{E}w \in H^1(\Omega)$), the application of Green's formula and the continuity of $\mathcal{N}^0 : \widetilde{H}^{-1}(\Omega) \to H^1(\Omega)$.

Outline

- Integral Representation Formula
- Newton Potential
- Single Layer Potential and Boundary Integral Operators associated to it
- Double Layer Potential and Boundary Integral Operators associated to it
- Calderón projector

The Single Layer Potential

- We have defined the **Single Layer Potential**

$$(\Psi^0_{\mathrm{SL}}\phi)(x) := \int_\Gamma \gamma_D G_0(x,y)\phi(y)\, dS(y), \quad x \in \mathbb{R}^d \backslash \Gamma \qquad (26)$$

  for functions $\phi \in L^1(\Gamma)$.
- But the domain of definition can be extended:

Theorem

The Single Layer Potential $\Psi^0_{SL} : H^{-1/2+s}(\Gamma) \to H^{1+s}(\Omega)$ *is bounded for* $|s| < 1/2$, *so,*

$$\|\Psi^0_{SL}\phi\|_{H^{1+s}(\Omega)} \leq c\|\phi\|_{H^{-1/2+s}(\Gamma)} \quad \forall \phi \in H^{-1/2+s}(\Gamma) \qquad (27)$$

The Single Layer Potential

- Furthermore,

Lemma

The function $u(x) := \Psi^0_{SL}\phi(x)$, $x \in \mathbb{R}^d \backslash \Gamma$ *is a solution of the homogeneous partial differential equation*

$$-\Delta u(x) = 0 \quad x \in \mathbb{R}^d \backslash \Gamma. \tag{28}$$

Proof.

Observe that $G_0(x, y)$ is C^∞ for $x \in \mathbb{R}^d \backslash \Gamma$ and $y \in \Gamma$. Therefore we can exchange integration and differentiation to obtain

$$\begin{aligned} -\Delta_x u(x) &= -\Delta_x \int_\Gamma G_0(x, y)\phi(y)dy \\ &= \int_\Gamma (-\Delta_x G_0(x, y))\phi(y)dy = 0. \end{aligned}$$

The Weakly Singular Boundary Integral Operator V_0

- $\Psi^0_{\mathrm{SL}}\phi \in H^{1+s}(\Omega)$ for $\phi \in H^{-1/2+s}(\Gamma)$, $|s| < 1/2$, so take the interior trace to obtain the **weakly singular boundary integral operator**

$$(V_0\phi)(x) := \gamma_D(\Psi^0_{\mathrm{SL}}\phi)(x) \quad \text{for } x \in \Gamma. \tag{29}$$

$V_0 : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma)$ is continuous, i.e. it satisfies

$$\|V_0\phi\|_{H^{1/2+s}(\Gamma)} \leq c\|\phi\|_{H^{-1/2+s}(\Gamma)} \quad \phi \in H^{1/2+s}(\Gamma) \tag{30}$$

Lemma

For $\phi \in L^\infty(\Gamma)$ we have the representation

$$(V_0\phi)(x) = \gamma_D(\Psi^0_{SL}\phi)(x) = \int_\Gamma G_0(x,y)\phi(y)dS(y) \quad x \in \Gamma \tag{31}$$

as a weakly singular surface integral.

The Weakly Singular Boundary Integral Operator V_0

- Similarly, we can take the exterior trace

$$(V_0\phi)(x) = \gamma_D^c(\Psi_{\mathrm{SL}}^0\phi)(x) := \lim_{\Omega^c \ni \widetilde{x} \to x \in \Gamma} (\Psi_{\mathrm{SL}}^0\phi)(\widetilde{x}) \quad \text{for } x \in \Gamma \qquad (32)$$

- We get the first **jump relation** of the single layer potential

$$[\gamma_D \Psi_{\mathrm{SL}}^0\phi] := \gamma_D^c(\Psi_{\mathrm{SL}}^0\phi)(x) - \gamma_D(\Psi_{\mathrm{SL}}^0\phi)(x) = 0 \quad x \in \Gamma \qquad (33)$$

for $\phi \in H^{-1/2+s}$.

- This follows from the properties of the trace operator and justifies that we denote the interior as well as the exterior trace of the single layer potential by the same symbol V_0.

Theorem

Let $\phi \in H^{-1/2}(\Gamma)$ for $d = 3$ and $\phi \in H_*^{-1/2}(\Gamma)$ for $d = 2$. Then we have

$$\langle V_0\phi, \phi\rangle_\Gamma \geq c\|\phi\|^2_{H^{-1/2}(\Gamma)} \tag{34}$$

for a positive constant c.

($H_*^{-1/2}(\Gamma) := \{\phi \in H^{-1/2}(\Gamma) : \langle \phi, 1\rangle_\Gamma = 0\}$.)

Proof.

1. application of interior and exterior Green's formulae $a_{\Omega^{(c)}}(u,u) = \left\langle \gamma_N^{(c)} u, \gamma_D^{(c)} u \right\rangle_\Gamma$.
2. application of jump relations, triangle and Cauchy-Schwarz inequality
3. a property of the weak solution $u \in H^1(\Omega)$ of the Dirichlet BVP $(-\Delta u)(x) = 0$, $x \in \Omega$, $\gamma_D u(x) = g(x)$, $x \in \Gamma$, namely $a(u,u) \geq c\|\gamma_N u\|^2_{H^{-1/2}(\Gamma)}$.

□

The Weakly Singular Boundary Integral Operator V_0

- Remark: For $\text{diam}(\Omega) < 1$ one can show that V_0 is $H^{-1/2}(\Gamma)$-elliptic for $d = 2$ as well, i.e.

$$\langle V_0 \phi, \phi \rangle_\Gamma \geq c \|\phi\|^2_{H^{-1/2}(\Gamma)} \quad \forall\, \phi \in H^{-1/2}(\Gamma) \tag{35}$$

- Notice that the $H^{-1/2}(\Gamma)$-ellipticity together with the $H^{-1/2}(\Gamma)$-continuity by the Lax-Milgram theorem imply that V_0 is invertible!

The Adjoint Double Layer Potential K_0'

- Apply the interior Neumann trace to the single layer potential of a function $\phi \in H^{-1/2+s}(\Gamma)$ ($|s| < 1/2$). Using (27) together with the properties of the Neumann trace operator γ_N, we get a bounded operator

$$\gamma_N \Psi_{\mathrm{SL}}^0 : H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma) \tag{36}$$

- Define the **adjoint double layer potential** K_0' by

$$(K_0'\phi)(x) := \lim_{\epsilon \to 0} \int_{y \in \Gamma : |y-x| \geq \epsilon} \gamma_{N,x} G_0(x,y)\phi(y) dS(y), \quad x \in \Gamma \tag{37}$$

and

$$\sigma(x) := \lim_{\epsilon \to 0} \frac{1}{2(d-1)\pi} \frac{1}{\epsilon^{d-1}} \int_{y \in \Omega : |y-x| = \epsilon} dS(y), \quad x \in \Gamma \tag{38}$$

- Remark: If Γ is differentiable in a neighborhood of $x \in \Gamma$, we have $\sigma(x) = \frac{1}{2}$.

Lemma

For $\phi \in H^{-1/2}(\Gamma)$, we have the representation

$$\gamma_N(\Psi_{SL}^0 \phi)(x) = \sigma(x)\phi(x) + (K_0' \phi)(x), \quad \textit{for } x \in \Gamma \tag{39}$$

in the sense of $H^{-1/2}(\Gamma)$, i.e.

$$\left\langle \gamma_N(\Psi_{SL}^0 \phi), \psi \right\rangle_\Gamma = \left\langle \sigma\phi + K_0'\phi, \psi \right\rangle_\Gamma, \quad \forall\, \psi \in H^{1/2}(\Gamma) \tag{40}$$

Proof:

1. Let $u := \Psi^0_{\mathsf{SL}}\phi$. Test with $\psi \in C^\infty(\Omega)$ and apply Green's first formula:

$$\left\langle \gamma_N(\Psi^0_{\mathsf{SL}}\phi), \psi \right\rangle_\Gamma = \int_\Omega \nabla_x u(x) \nabla_x \psi(x) dx$$

2. Insert the definition of the single layer and write as a limit

$$\begin{aligned} \int_\Omega \nabla_x u(x) \nabla_x \psi(x) dx &= \int_\Omega \nabla_x \int_\Gamma G_0(x,y)\phi(y) dS(y) \nabla_x \psi(x) dx \\ &= \int_\Omega \nabla_x \left(\lim_{\epsilon \to 0} \int_{y \in \Gamma : |x-y| \geq \epsilon} G_0(x,y)\phi(y) dS(y) \right) \nabla_x \psi(x) dx \end{aligned} \tag{41}$$

- Interchange the integral signs, integrate by parts and apply Green's formula again to the interior integral:

$$\begin{aligned}
&\int_{x\in\Omega:|x-y|\geq\epsilon} \nabla_x G_0(x,y)\nabla_x\psi(x)dx \\
&= \int_{x\in\Gamma:|x-y|\geq\epsilon} \gamma_{N,x} G_0(x,y)\gamma_D\psi(x)dS(x) \\
&+ \int_{x\in\Omega:|x-y|=\epsilon} \gamma_{N,x} G_0(x,y)\psi(x)dS(x) \qquad (42)
\end{aligned}$$

- Estimate the obtained summands separately, by explicitely inserting the formula for $\gamma_{N,x}G_0(x,y)$ in $d=2,3$.

The Adjoint Double Layer Potential K_0'

- If we go through the proof of the last lemma, we find for the exterior Neumann trace of the single layer potential the following representation:

$$\gamma_N^c(\Psi_{SL}^0 \phi)(x) = (\sigma(x) - 1)\phi(x) + (K_0'\phi)(x), \quad \text{for } x \in \Gamma \tag{43}$$

  in the sense of $H^{-1/2}(\Gamma)$.

- We obtain the second **jump relation** of the single layer potential:

$$[\gamma_N \Psi_{SL}^0 \phi] := \gamma_N^c(\Psi_{SL}^0 \phi)(x) - \gamma_N(\Psi_{SL}^0 \phi)(x) = -\phi(x), \quad \text{for } x \in \Gamma, \tag{44}$$

  in the sense of $H^{-1/2}(\Gamma)$.

Outline

- Integral Representation Formula
- Newton Potential
- Single Layer Potential and Boundary Integral Operators associated to it
- Double Layer Potential and Boundary Integral Operators associated to it
- Calderón projector

The Double Layer Potential

- The **double layer potential**

$$(\Psi^0_{\mathrm{DL}}\phi)(x) := \int_\Gamma \gamma_{N,y} G_0(x,y)\phi(y)\,dS(y), \quad x \in \mathbb{R}^d \backslash \Gamma \tag{45}$$

defines a continuous operator $\Psi^0_{\mathrm{DL}} : H^{1/2+s}(\Gamma) \to H^{1+s}(\Omega)$ for $|s| < 1/2$, i.e.

$$\|\Psi^0_{\mathrm{DL}}\phi\|_{H^{1+s}(\Omega)} \leq c\|\phi\|_{H^{1/2+s}(\Gamma)} \quad \forall \phi \in H^{1/2+s}(\Gamma). \tag{46}$$

- As in the case of the single layer potential, $u \in H^1(\Omega)$ defined by $u(x) = (\Psi^0_{\mathrm{DL}}\phi)(x)$, $x \in \mathbb{R}^d \backslash \Gamma$ for $\psi \in H^{1/2}(\Gamma)$ is a solution to the homogeneous partial differential equation

$$-\Delta_x u(x) = 0, \quad x \in \mathbb{R}^d \backslash \Gamma. \tag{47}$$

The Boundary Integral Operator K_0

- Apply the interior trace operator to the double layer potential to obtain a bounded linear operator

$$\gamma_D \Psi^0_{\mathrm{DL}} : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \tag{48}$$

  i.e.

$$\|\gamma_D \Psi^0_{\mathrm{DL}} \phi\|_{H^{1/2+s}(\Gamma)} \leq c \|\phi\|_{H^{1/2+s}(\Gamma)} \quad \forall \phi \in H^{1/2+s}(\Gamma) \tag{49}$$

- Define for $\phi \in H^{1/2+s}(\Gamma)$ the **boundary integral operator** K_0 by

$$(K_0 \phi)(x) := \lim_{\epsilon \to 0} \int_{y \in \Gamma : |y-x| \geq \epsilon} (\gamma_{N,y} G_0(x,y)) \phi(y) dS(y). \tag{50}$$

The Boundary Integral Operator K_0

Lemma

For $\phi \in H^{1/2}(\Gamma)$ we have the representation

$$\gamma_D(\Psi^0_{DL}\phi)(x) = (-1 + \sigma(x))\phi(x) + (K_0\phi)(x) \quad \textit{for } x \in \Gamma, \tag{51}$$

where σ is defined in (38). For the exterior trace we have the representation

$$\gamma^c_D(\Psi^0_{DL}\phi)(x) = \sigma(x)\phi(x) + (K_0\phi)(x) \quad \textit{for } x \in \Gamma, \tag{52}$$

Hence, we obtain the first **jump relation** of the double layer potential:

$$[\gamma_D\Psi^0_{\mathrm{DL}}\phi] := \gamma^c_D(\Psi^0_{\mathrm{DL}}\phi)(x) - \gamma_D(\Psi^0_{\mathrm{DL}}\phi)(x) = \phi(x) \quad \text{for } x \in \Gamma. \tag{53}$$

The Hypersingular Boundary Integral Operator W_0

- For $\phi \in H^{1/2+s}(\Gamma)$, $|s| < 1/2$, we define for $x \in \Gamma$:

$$(W_0\phi)(x) := -\gamma_N(\Psi^0_{\mathrm{DL}}\phi)(x) = -\lim_{\Omega \ni \widetilde{x} \to x \in \Gamma} n_x \cdot \nabla_{\widetilde{x}}(\Psi^0_{\mathrm{DL}}\phi)(\widetilde{x}) \quad (54)$$

  W_0 is the **hypersingular boundary integral operator**.
- $W_0 : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma)$ is bounded for $|s| < 1/2$, so

$$\|W_0\phi\|_{H^{-1/2+s}(\Gamma)} \leq c\|\phi\|_{H^{1/2+s}(\Gamma)} \quad \forall\, \phi \in H^{1/2+s}(\Gamma). \quad (55)$$

- Associated to the hypersingular boundary integral operator is a bilinear form:

$$\langle W_0 v, w\rangle_\Gamma = -\int_\Gamma w(x)\gamma_{N,x}\int_\Gamma \gamma_{N,y}G_0(x,y)v(y)dS(y)dS(x). \quad (56)$$

The Hypersingular Boundary Integral Operator W_0

- Insert $u_0(x) = 1$ into the representation formula (9) to obtain

$$1 = -\int_\Gamma \gamma_{N,y} G_0(\widetilde{x}, y) dS(y), \quad \widetilde{x} \in \Omega.$$

- Therefore

$$\nabla_{\widetilde{x}}(\Psi^0_{\mathrm{DL}} u_0)(\widetilde{x}) = 0, \quad \widetilde{x} \in \Omega.$$

- Taking the limit $\widetilde{x} \to x \in \Gamma$

$$(W_0 u_0)(x) = 0, \quad x \in \Gamma.$$

- This implies that we cannot get ellipticity of W_0 on $H^{1/2}(\Gamma)$. So we need to consider a subspace:

The Hypersingular Boundary Integral Operator W_0

- Consider in $d = 2, 3$ the problem: Find $(t, \lambda) \in H^{-1/2}(\Gamma) \times \mathbb{R}$ such that

$$\begin{aligned} \langle V_0 t, \tau \rangle_\Gamma - \lambda \langle 1, \tau \rangle_\Gamma &= 0 \qquad \forall \tau \in H^{-1/2}(\Gamma) \\ \langle t, 1 \rangle_\Gamma &= 1 \qquad . \end{aligned} \tag{57}$$

- Using the ansatz $t := \widetilde{t} + 1/|\Gamma|$ for arbitrary $\widetilde{t} \in H_*^{-1/2}(\Gamma)$, the second equation is already satisfied and the first one reads

$$\left\langle V_0 \widetilde{t}, \tau \right\rangle_\Gamma = -\frac{1}{|\Gamma|} \langle V_0 1, \tau \rangle_\Gamma \quad \forall \tau \in H_*^{-1/2}(\Gamma) \tag{58}$$

- This variational problem is uniquely solvable which follows from the $H_*^{-1/2}(\Gamma)$-ellipticity of the operator V_0. The solution $w_{eq} := \widetilde{t} + 1/|\Gamma|$ is denoted as the **natural density**. We have $\lambda = \langle V_0 w_{eq}, w_{eq} \rangle_\Gamma$.

The Hypersingular Boundary Integral Operator W_0

- Now we introduce the space

$$H_*^{1/2}(\Gamma) := \left\{ v \in H^{1/2}(\Gamma) : \langle v, w_{\text{eq}} \rangle_\Gamma = 0 \right\}.$$

(One can show then that $V_0 : H_*^{-1/2}(\Gamma) \to H_*^{1/2}(\Gamma)$ is an isomorphism.)

Theorem

The hypersingular operator is $H_*^{1/2}(\Gamma)$-elliptic, i.e.

$$\langle W_0 \phi, \phi \rangle_\Gamma \geq c \|\phi\|^2_{H^{1/2}(\Gamma)} \quad \forall \, \phi \in H_*^{1/2}(\Gamma). \tag{59}$$

The proof uses the representation of the trace operators (51), (52), (47), Green's formulae, the jump relations, the norm equivalence theorem of Sobolev and the trace theorem.

The Hypersingular Boundary Integral Operator W_0

- Finally, we have the second **jump relation** of the double layer potential:

$$[\gamma_N \Psi^0_{\mathrm{DL}} \phi] := \gamma^c_N(\Psi^0_{\mathrm{DL}} \phi)(x) - \gamma_N(\Psi^0_{\mathrm{DL}} \phi)(x) = 0, \quad x \in \Gamma. \tag{60}$$

Summary

Summing up, we have

Theorem

Let Ω be a Lipschitz domain, $\partial\Omega =: \Gamma$ its boundary. Then for all $s \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ the following boundary operators are bounded:

$$V_0 : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \tag{61}$$

$$K_0 : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \tag{62}$$

$$K_0' : H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma), \tag{63}$$

$$W_0 : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma), \tag{64}$$

and the **jump relations** for $\phi \in H^{-1/2}(\Gamma)$ and $\phi \in H^{1/2}(\Gamma)$

$$[\gamma_D \Psi^0_{\mathrm{SL}} \phi] = 0, \qquad [\gamma_D \Psi^0_{\mathrm{DL}} \psi] = \psi, \qquad \text{in } H^{1/2}(\Gamma) \tag{65}$$

$$[\gamma_N \Psi^0_{\mathrm{SL}} \phi] = -\phi, \qquad [\gamma_N \Psi^0_{\mathrm{DL}} \psi] = 0, \qquad \text{in } H^{-1/2}(\Gamma) \tag{66}$$

Outline

- Integral Representation Formula
- Newton Potential
- Single Layer Potential and Boundary Integral Operators associated to it
- Double Layer Potential and Boundary Integral Operators associated to it
- Calderón projector

Calderón projector

- Consider again representation formula (9).

$$u(x) = (\mathcal{N}^0 f)(x) - \Psi^0_{\mathrm{DL}}(\gamma_D u)(x) + \Psi^0_{\mathrm{SL}}(\gamma_N u)(x) \quad \text{for } x \in \mathbb{R}^d \backslash \Gamma. \tag{67}$$

- Take Dirichlet and Neumann trace of the representation formula:

$$\gamma_D u(x) = (V_0 \gamma_N u)(x) + (1 - \sigma(x))\gamma_D u(x) - (K_0 \gamma_D u)(x) + \gamma_D \mathcal{N}^0 f(x), \tag{68}$$

and

$$\gamma_N u(x) = \sigma(x)\gamma_N u(x) + (K_0' \gamma_N u)(x) + (W_0 \gamma_D u)(x) + \gamma_N \mathcal{N}^0 f(x), \tag{69}$$

for $x \in \Gamma$.

Calderón projector

- We have obtained a system of boundary integral equations

$$\begin{pmatrix}\gamma_D u\\ \gamma_N u\end{pmatrix} = \begin{pmatrix}\frac{1}{2}I - K_0 & V_0\\ W_0 & \frac{1}{2}I + K_0'\end{pmatrix}\begin{pmatrix}\gamma_D u\\ \gamma_N u\end{pmatrix} + \begin{pmatrix}\gamma_D \mathcal{N}^0 f\\ \gamma_N \mathcal{N}^0 f\end{pmatrix} \tag{70}$$

where

$$\mathcal{C} := \begin{pmatrix}\frac{1}{2}I - K_0 & V_0\\ W_0 & \frac{1}{2}I + K_0'\end{pmatrix} \tag{71}$$

is the **Calderón projector**.

Lemma

The operator $\mathcal{C}$ as defined in (71) is a projection, i.e. $\mathcal{C} = \mathcal{C}^2$.

Calderón projector

Proof.

1. take $(v, w) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Then $u(\widetilde{x}) := (\Psi^0_{\mathrm{SL}} v)(\widetilde{x}) - (\Psi^0_{\mathrm{DL}} w)(\widetilde{x})$ for $\widetilde{x} \in \Omega$ is a solution of the homogeneous partial differential equation.
2. For the traces of u we find:

$$
\begin{aligned}
\gamma_D u(x) &= (V_0 v)(x) + \frac{1}{2} w(x) - (K_0 w)(x), \\
\gamma_N u(x) &= \frac{1}{2} v(x) + (K_0' v)(x) + (W_0 w)(x).
\end{aligned}
\tag{72}
$$

3. So we have found the Cauchy data $(\gamma_D u(x), \gamma_N u(x))$, which satify (68) and (69). This is equivalent to (70) with $f \equiv 0$.
4. Insert (72) for $(\gamma_D u(x), \gamma_N u(x))^T$ to obtain the claim.

□

Half past six slide

Corollary

For the boundary integral operators, we have the relations

$$V_0 W_0 = \left(\frac{1}{2}I + K_0\right)\left(\frac{1}{2}I - K_0\right), \tag{73}$$

$$W_0 V_0 = \left(\frac{1}{2}I + K_0'\right)\left(\frac{1}{2}I - K_0'\right), \tag{74}$$

$$V_0 K_0' = K_0 V_0, \tag{75}$$

$$K_0' W_0 = W_0 K_0. \tag{76}$$

Since we have found that V_0 and W_0 are invertible operators (if we consider the correct subspaces of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$) this allows us to find the complete Cauchy data if we have either $\gamma_D u(x)$ or $\gamma_N u(x)$ and hence find the solution to the Laplace equation.

Questions?

Thanks you for listening to me.

Aufwachen!