Sobolev spaces, Trace theorems and Green's functions.

Boundary Element Methods for Waves Scattering Numerical Analysis Seminar.

Orane Jecker

October 21, 2010

Plan

Introduction

Main subjects

Definition

Let $d \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ be a multi index with absolute value $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_d$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. For u a real valued function which is sufficiently smooth, the partial derivative is given by

$$D^{\boldsymbol{\alpha}} u(\mathbf{x}) := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} \cdot u(x_1, \ldots, x_d).$$

- Let $k \in \mathbb{N}_0$ and $\kappa \in (0,1)$.

$$C^{k,\kappa}(\Omega) := \{u : \Omega \to \mathbb{R} \mid D^k u \text{ is Hölder continuous with exponent } \kappa\}.$$

- The associated norm is

$$\|u\|_{C^{k,\kappa}(\Omega)} := \|u\|_{C^k(\Omega)} + \sum_{|\alpha|=k} \sup_{x,y\in\Omega, x\neq y} \frac{| D^{|\alpha|}u(x) - D^{|\alpha|}u(y) |}{| x-y |^{\kappa}}$$

Simplest case:
There exists a function $\gamma : \mathbb{R}^{d-1} \longrightarrow \mathbb{R}$ such that

$$\Omega := \{x \in \mathbb{R}^d \mid x_d < \gamma(\tilde{x}) \text{ for all } \tilde{x} = (x_1, \dots, x_{d-1}) \in \mathbb{R}^{d-1}\}.$$

Definition

When γ is Lipschitz, then Ω is said to be a Lipschitz hypograph with boundary

$$\partial\Omega =: \Gamma := \{x \in \mathbb{R}^d \mid x_d = \gamma(\tilde{x}) \text{ for all } \tilde{x} \in \mathbb{R}^{d-1}\}.$$

Definition

An open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$ is a Lipschitz domain if Γ is compact and if there exist finite families $\{W_i\}$ and $\{\Omega_i\}$ such that:

1. $\{W_i\}$ is a finite open cover of Γ, that is $W_i \subset \mathbb{R}^d$ is open for all $i \in \mathbb{N}$ and $\Gamma \subseteq \cup_i W_i$.
2. Each Ω_i can be transformed into a Lipschitz hypograph by a rigid motion
3. For all $i \in \mathbb{N}$ the equality $W_i \cap \Omega = W_i \cap \Omega_i$.

- The local representation of the boundary Γ is in general not unique.

Example of a non-lipschitz domain

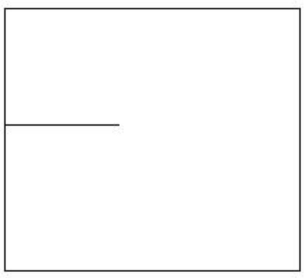


Figure: Example of a non-lipschitz domain in 2D.

Definition

$$L_1^{loc} := \{u : \Omega \rightarrow \mathbb{R} \mid \text{ u is locally integrable } \}.$$

That means, u is integrable with respect to any bounded closed subset K of Ω.

Remark

A function $u : \Omega \rightarrow \mathbb{R} \in L_1^{loc}(\Omega)$ is not, in general, in $L_1(\Omega)$. On the other hand, $u \in L_1(\Omega)$ implies that $u \in L_1^{loc}(\Omega)$.

$$\int_\Omega u(x)\mathrm{dx} < \infty \Rightarrow \int_K u(x)\mathrm{dx} < \infty, \quad \forall K \subseteq \Omega.$$

Example

Let $\Omega = (0,1)$ and $u(x) = \frac{1}{x}$. We have

$$\int_0^1 u(x)dx = \lim_{\epsilon \longrightarrow 0} \int_\epsilon^1 \frac{1}{x}\mathrm{dx} = \lim_{\epsilon \longrightarrow 0} \ln\left(\frac{1}{\epsilon}\right) = \infty.$$

- That implies $u \notin L_1(\Omega)$.
  Let $K = [a,b] \subset (0,1)$ with $0 < a < b < 1$. Then

$$\int_K u(x)\mathrm{dx} = \int_a^b \frac{1}{x}\mathrm{dx} = \ln\left(\frac{b}{a}\right) < \infty.$$

- That implies $u \in L_1^{\text{loc}}(\Omega)$.

Definition

A function $u \in L_1^{\text{loc}}(\Omega)$ has a generalized partial derivative w.r.t. x_i, if there exists $v \in L_1^{\text{loc}}(\Omega)$ such that

$$\int\limits_{\Omega} v(x)\varphi(x)\mathrm{d}x = -\int\limits_{\Omega} u(x)\frac{\partial}{\partial x_i}\varphi(x)\mathrm{d}x, \text{ for all } \varphi \in C_0^\infty(\Omega).$$

The GPD is denoted by $\frac{\partial}{\partial x_i}u(x) := v(x)$.

We define the space of test functions by $C_0^\infty(\Omega) := \mathscr{D}(\Omega)$.

Definition

Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \Omega$.
$\{\varphi_n\}$ converges to φ in $\mathscr{D}(\Omega)$ if

1. $\exists K \subset \Omega$ compact subset such that $\operatorname{supp}\varphi_n \subset K, \forall n \in \mathbb{N}$
2. $D^{\boldsymbol{\alpha}}\varphi_n \xrightarrow[\|\cdot\|_{C_0^\infty}]{} D^{\boldsymbol{\alpha}}\varphi, \forall \boldsymbol{\alpha} \in \mathbb{N}^d$.

Definition

A complex valued continuous linear map $T : \mathscr{D}(\Omega) \to \mathbb{C}$ is called a distribution. T is continuous if

$$\lim_{n\to\infty} T(\varphi_n) = T(\varphi),$$

for any $\{\varphi_n\}_{n\in\mathbb{N}}$ which converges to φ in $\mathscr{D}(\Omega)$.
The set of all distributions is denoted by $\mathscr{D}'(\Omega)$.

Definition

Let $T \in \mathscr{D}'(\Omega)$. Its partial derivative w.r.t x_i, $1 \leq i \leq d$, in the sense of distribution is

$$\partial_i T(\varphi) = -T(\partial_i \varphi), \text{ for all } \varphi \in \mathscr{D}(\Omega)$$

Definition

For a function $u \in L_1^{\text{loc}}(\Omega)$ we define the distribution

$$T_u(\varphi) := \int_\Omega u(x)\varphi(x)\text{dx}, \text{ for } \varphi \in \mathscr{D}(\Omega).$$

Example

Let $v(x) = \text{sign}(x) \in L_1^{\text{loc}}([-1, 1])$ and compute its derivative in the sense of distribution.

$$\int_{-1}^{1} \frac{\partial}{\partial x}\text{sign}(x)\varphi(x)\text{dx} = -\int_{-1}^{1} \text{sign}(x)\frac{\partial}{\partial x}\varphi(x)\text{dx} = 2\varphi(0),$$

for all $\varphi \in \mathscr{D}(\Omega)$.
We obtain

$$\frac{\partial}{\partial x}\text{sign}(x) = 2\delta_0 \in \mathscr{D}'(\Omega).$$

SOBOLEV SPACES

Definition

Let $k \in \mathbb{N}_0$, the *Sobolev space* is defined as

$$W_p^k(\Omega) := \overline{C^\infty(\Omega)}^{\|\cdot\|_{W_p^k(\Omega)}}.$$

The norm is given by

$$\|u\|_{W_p^k(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L_p(\Omega)}\right)^{\frac{1}{p}}, & \text{for } 1 \le p < \infty, \\ max_{|\alpha|\le k} \|D^\alpha u\|_{L_\infty(\Omega)}, & \text{for } p = \infty. \end{cases}$$

$$\mathring{W}_p^k(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W_p^k(\Omega)}}.$$

Sobolev spaces can be define for all $s \in \mathbb{R}$.

- For $0 < s$, with $s = k + \kappa$, $k \in \mathbb{N}_0$ and $\kappa \in (0,1)$, the norm is

$$\|u\|_{W_p^s(\Omega)} := \left(\|u\|^p_{W_p^k(\Omega)} + |u|^p_{W_p^s(\Omega)} \right)^{\frac{1}{p}},$$

where

$$|u|^p_{W_p^s(\Omega)} = \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha (y)|^p}{|x-y|^{d+p\kappa}} \mathrm{dxdy}$$

- For $s < 0$ and $1 < p < \infty$, $W_p^s(\Omega) := \left(\mathring{W}_q^{-s}(\Omega)\right)'$ where $\frac{1}{p} + \frac{1}{q} = 1$. The norm is

$$\|u\|_{W_p^s(\Omega)} := \sup_{v \in \mathring{W}_q^{-s}(\Omega), v \neq 0} \frac{|\langle u, v \rangle_\Omega|}{\|v\|_{W_q^{-s}(\Omega)}}$$

The Sobolev space $W_2^s(\Omega)$ admits an inner-product

- For $s = k \in \mathbb{N}_0$

$$\langle u, v\rangle_{W_2^k(\Omega)} := \sum_{|\alpha|\le k} \int_\Omega D^\alpha u(x) D^\alpha v(x) \mathrm{d}x$$

- For $s = k + \kappa$ with $\kappa \in (0,1)$ and $k \in \mathbb{N}_0$

$$\langle u, v\rangle_{W_2^s(\Omega)} := \langle u, v\rangle_{W_2^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x-y|^{d+2\kappa}} \mathrm{d}x\mathrm{d}y$$

Definition

We define the space of rapidly decreasing functions

$$\mathcal{S}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \|\varphi\|_{k,l} < \infty\},$$

where

$$\|\varphi\|_{k,l} = \sup_{x \in \mathbb{R}^d} (|x|^k + 1) \sum_{|\alpha| \leq l} |D^\alpha \varphi(x)| < \infty, \text{ for all } k, l \in \mathbb{N}_0.$$

The space of *tempered distributions* $\mathcal{S}'(\mathbb{R}^d)$ is

$$\mathcal{S}'(\mathbb{R}^d) := \{T : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} \mid T \text{ complex valued cont. lin. map}\}.$$

- For $s \in \mathbb{R}$, the Bessel potential operator $\mathcal{J}^s : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$\mathcal{J}^s u(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi) e^{i\langle x, \xi \rangle} d\xi,$$

for $u \in \mathcal{S}(\mathbb{R}^d)$.

$$(\mathcal{J}^s T)(\varphi) := T(\mathcal{J}^s \varphi), \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Definition

The Sobolev space over $\mathbb{R}^d$ is defined as

$$H^s(\mathbb{R}^d) := \{v \in \mathcal{S}'(\mathbb{R}^d) \mid \mathcal{J}^s v \in L_2(\mathbb{R}^d)\}, \text{ for all } s \in \mathbb{R}.$$

The norm is

$$\|v\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 \mathrm{d}\xi.$$

Theorem

For all $s \in \mathbb{R}$, we have the following relation

$$H^s(\mathbb{R}^d) = W_2^s(\mathbb{R}^d).$$

Let Ω be a bounded domain in $\mathbb{R}^d$,

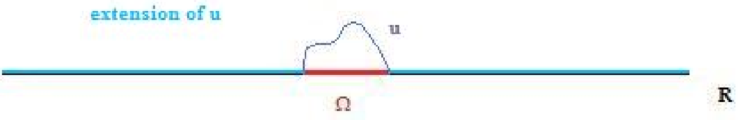
$$H^s(\Omega) := \{v = \tilde{v}_{|\Omega} \mid \tilde{v} \in H^s(\mathbb{R}^d)\},$$

the norm is given by

$$\|v\|_{H^s(\Omega)} := \inf_{\tilde{v} \in H^s(\mathbb{R}^d), \tilde{v}_{|\Omega} = v} \|\tilde{v}\|_{H^s(\mathbb{R}^d)}.$$

Definition

$$\tilde{H}^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^d)}}, \ H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}$$



Theorem

Let $\Omega \in \mathbb{R}^d$ be a Lipschitz domain. For $s \geq 0$ we have

$$\tilde{H}^s(\Omega) \subset H_0^s(\Omega).$$

Moreover,

$$\tilde{H}^s(\Omega) = H_0^s(\Omega) \text{ for } s \notin \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}.$$

$$\tilde{H}^s(\Omega) = [H^{-s}(\Omega)]', H^s(\Omega) = [\tilde{H}^{-s}(\Omega)]'$$

For $s < 0$, we define $H^s(\Gamma) := (H^{-s}(\Gamma))'$ with the norm

$$\|u\|_{H^s(\Gamma)} := \sup_{0 \neq v \in H^{-s}(\Gamma)} \frac{\langle u, v \rangle_\Gamma}{\|v\|_{H^{-s}(\Gamma)}}$$

Definition

The Sobolev spaces over a bounded domain $\Omega \in \mathbb{R}^d$ allow us to define the Fréchet space

$$H^1_{\text{loc}}(\Omega) := \{v \in \mathscr{D}'(\Omega) \mid \|v\|_{H^1(B)} < +\infty \quad \text{for all bounded } B \subset \Omega\},$$

and the space

$$H^1_{\text{comp}}(\Omega) := \{v \in \mathscr{D}'(\Omega) \mid v \in H^1(\Omega), \text{v has compact support}\}$$

Remark

We see that $H^1_{\text{comp}}(\Omega) \subset H^1_{\text{loc}}(\Omega)$.

Theorem (Sobolev Embedding Theorem.)

Let Ω be an open subset of $\mathbb{R}^d$ with a lipschitz continuous boundary. The following continuous embedding holds

- *For all* $s \in \mathbb{R}$, $H^{s+1}(\Omega) \subset H^s(\Omega)$.
- *For* $k \in \mathbb{N}_0$, *if* $2(k-m) > d$, *then* $H^k(\Omega) \subset C^m(\bar{\Omega})$.
- *For all* $s \in \mathbb{R}$, *if*

$$d \leq s \text{ for } p = 1, \quad \frac{d}{p} < s \text{ for } p > 1$$

then $W_p^s(\Omega) \subset C(\Omega)$

Our goal is to study the continuity of function in Sobolev space.

Take $\mathbf{m} = \mathbf{0}$.

1. If $\mathbf{d} = \mathbf{1}$, $2(k-0) > 1$ is valid $\forall k \geq 1$.
   - We have $H^1(\Omega) \subset C^0(\bar{\Omega})$.
   - Moreover, as $H^{k+1}(\Omega) \subset H^k(\Omega)$,

$$H^{k+1}(\Omega) \subset H^k(\Omega) \subset \ldots \subset H^1(\Omega) \subset C^0(\bar{\Omega}).$$

2. If $\mathbf{d} = \mathbf{2}$, is $H^1(\Omega) \subset C^0(\bar{\Omega})$?
   - No,

$$H^1(\Omega) \not\subset C^0(\bar{\Omega})!$$

   - We have

$$H^2(\Omega) \subset C^0(\bar{\Omega}).$$

TRACE THEOREMS

- We recall the interior trace operator
  $\gamma_D := \gamma_0^{int} : C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$.

Theorem (Trace Theorem.)

Let Ω be a $C^{k-1,1}$-domain. For $\frac{1}{2} < s \leq k$ the interior trace operator

$$\gamma_D : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma),$$

where $\gamma_D v := v_{|\Gamma}$, is bounded. There exists $C_T > 0$ such that

$$\|\gamma_D v\|_{H^{s-\frac{1}{2}}(\Gamma)} \leq C_T \|v\|_{H^s(\Omega)} \text{ for all } v \in H^s(\Omega).$$

- For a lipschitz domain Ω, put $k = 1$.
  $\gamma_D : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Omega)$ for $s \in (\frac{1}{2}, 1]$. That is true for $s \in (\frac{1}{2}, \frac{3}{2})$

Theorem (Inverse Trace Theorem.)

The trace operator $\gamma_D : H^s(\Omega) \to H^{s-\frac{1}{2}}(\Gamma)$ *has a continuous right inverse operator*

$$\mathcal{E} : H^{s-\frac{1}{2}}(\Gamma) \to H^s(\Omega)$$

satisfying $(\gamma_D \circ \mathcal{E})(w) = w$ *for all* $w \in H^{s-\frac{1}{2}}(\Gamma)$. *There exists a constant* $C_I > 0$ *such that*

$$\|\mathcal{E}w\|_{H^s(\Omega)} \leq C_I \|w\|_{H^{s-\frac{1}{2}}(\Gamma)}, \quad \textit{for all } w \in H^{s-\frac{1}{2}}(\Gamma).$$

- $\gamma_D : H^s(\Omega) \longrightarrow H^{s-\frac{1}{2}}(\Gamma)$ is surjective and its continuous right inverse $\mathcal{E} : H^{s-\frac{1}{2}}(\Gamma) \longrightarrow H^s(\Omega)$ is injective.
- With the two last Theorems, we can redefine the Sobolev space $H^s(\Gamma)$.
- For $s > 0$. $H^s(\Gamma)$ can be seen as the space of traces of $H^{s+\frac{1}{2}}(\Omega)$.
- The interest of fractional Sobolev spaces comes from the use of the Green's theorems in BEM.

GREEN'S FORMULA AND FUNCTIONS - Fundamental solutions

Theorem

1. *For* $u \in H^2(\Omega)$ *and* $v \in H^1(\Omega)$, *the first Green's formula is*
$$\langle \Delta u, v\rangle_\Omega = -\langle \nabla u, \nabla v\rangle_\Omega + \langle \gamma_D \frac{\partial u}{\partial \mathbf{n}}, \gamma_D v\rangle_{\partial\Omega}.$$

2. *For* $u, v \in H^2(\Omega)$, *the second Green's formula is*
$$\langle \Delta u, v\rangle_\Omega - \langle \Delta v, u\rangle_\Omega = \langle \gamma_D \frac{\partial u}{\partial \mathbf{n}}, \gamma_D v\rangle_{\partial\Omega} - \langle \gamma_D \frac{\partial v}{\partial \mathbf{n}}, \gamma_D u\rangle_{\partial\Omega}.$$

Let us consider a scalar partial differential equation

$$(\mathcal{L}u)(x) = f(x), x \in \Omega \subset \mathbb{R}^d.$$

Definition

A fundamental solution of the PDE is the solution of

$$(\mathcal{L}_y G(x,y))(x,y) = \delta_0(y-x), x, y \in \mathbb{R}^d,$$

in the distributional sense.

Green's function of a PDE is a fundamental solution satisfying the boundary conditions.
Green's functions are distributions.

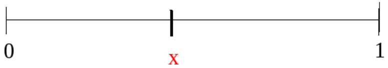
Example

Compute the Green's function $G(x, y)$ such that

$$u(x) = \int_0^1 G(x, y) f(y) \mathrm{d}y, \ \text{ for } x \in (0, 1),$$

is the unique solution of the Dirichelet BVP

$$\begin{cases} -u''(x) = f(x), \ \text{ for } x \in (0, 1) \\ u(0) = u(1) = 0. \end{cases}$$



- Solve $-G''(x,y) = \delta(x,y)$ and split $(0,1)$ into $0 < y < x$ and $x < y < 1$.
- $-G'' = 0$ on each side

$$\Rightarrow G(x,y) = \begin{cases} a_1(x)y + b_1(x), y \in (0,x) \\ a_2(x)y + b_2(x), y \in (x,1) \end{cases}$$

- Boundary conditions $\Rightarrow b_1(x) = 0$ and $b_2(x) = -a_2(x)$.
- $\forall \varphi \in \mathscr{D}([0,1])$ solve

$$\langle -G''(x,\cdot), \varphi \rangle = \varphi(x) \Rightarrow \langle G(x,\cdot), \varphi'' \rangle = -\varphi(x).$$

-

$$G(x,y) = \begin{cases} (1-x)y, y \in (0,x) \\ x(1-y), y \in (x,1) \end{cases}$$

- Let us consider the Laplace operator

$$(\mathcal{L}u)(x) := -\Delta u(x) \text{ for } x \in \mathbb{R}^d, d = 2, 3.$$

- The fundamental solution $G(x, y)$ is the distributional solution of the PDE

$$-\Delta_y G(x, y) = \delta_0(y - x) \text{ for } x, y \in \mathbb{R}^d.$$

- First put

$$G(x, y) = v(z), \text{ where } z = y - x.$$

- Hence solve

$$-\Delta v(z) = \delta_0(z), \; z \in \mathbb{R}^d.$$

- We apply the Fourier transformation and obtain

$$\hat{v}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{1}{|\xi|^2} \in \mathcal{S}'(\mathbb{R}^d)$$

Definition

For a distribution $T \in \mathcal{S}'(\mathbb{R}^d)$, the Fourier transform is given by

$$\hat{T}(\varphi) = T(\hat{\varphi}), \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^d).$$

- Hence, we have to solve

$$\langle \hat{v}, \varphi \rangle_{L^2(\mathbb{R}^d)} = \langle v, \hat{\varphi} \rangle_{L^2(\mathbb{R}^d)}, \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^d).$$

For $d = 3$.

- The fundamental solution is

$$G(x, y) = \frac{1}{4\pi} \frac{1}{|x - y|}, x, y \in \mathbb{R}^3.$$

- $G(x, y)$ is continuous (C^∞) except when $x = y$.
- The fundamental solution is bounded at the infinity.

$$\text{When } x \text{ or } y \to \infty \Rightarrow G(x, y) \to 0.$$

- $G(x, y) \notin L^2(\mathbb{R}^3)$. Consider $v(z) = \frac{1}{z}$

$$\|v(z)\|_{L^2(\mathbb{R}^3)} = \int_{-\infty}^{\infty} \frac{1}{z^2} \mathrm{dz} = \underbrace{\int_{-\infty}^{0} \frac{1}{z^2} \mathrm{dz}}_{\to\infty} + \underbrace{\int_{0}^{\infty} \frac{1}{z^2} \mathrm{dz}}_{\to\infty} \to \infty$$

For $d = 2$.

- The fundamental solution is

$$G(x,y) = -\frac{1}{2\pi}\log(|x-y|), x, y \in \mathbb{R}^2.$$

- $G(x,y) \in C^\infty$ except on $x = y$.
- The fundamental solution is unbounded at the infinity.

$$\text{When } x \text{ or } y \to \infty \Rightarrow U(x,y) \to \infty.$$

- The problem of at the infinity is solved with the boundary conditions in the Green function.

The fundamental solution for the Helmolzt equation

$$-\Delta u(x) - 2ku(x) = 0 \text{ for } x \in \mathbb{R}^d, k \in \mathbb{R},$$

is

- for $d = 3$

$$G_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, x, y \in \mathbb{R}^3.$$

- for $d = 2$

$$G_k(x,y) = \frac{1}{2\pi} Y_0(k|x-y|), x, y \in \mathbb{R}^3,$$

where $Y_0 =$ second Bessel function of order zero.