Matrix construction: Singular integral contributions

Seminar Boundary Element Methods for Wave Scattering

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November 2010

Outline

Weakly singular integrals

Definition

A singular integral is said to be *weakly singular* if its value exists and is continuous at the singularity point.

Example

Consider $[0, a] \subset \mathbb{R}$

$$\int_0^a ln|x| = (xln|x| - x)\Big|_0^a$$

Using a limit approach and the rule of De l' Hopital for the first term on the RHS, we can easily find, that the integral is continuous at 0, although the function is singular at 0.

Strongly singular integrals

Definition

An integral

$$\int f(x)dx$$

is said to be *strongly singular* if not only the integrand f(x) is singular at a point c, but also the integral itself is singular at that point, too.

Example

$$\text{in 2-D: } \frac{1}{r} \qquad \text{in 3-D: } \frac{1}{r^3}$$

When looking at strongly singular integrals, we can interpret their value in terms of the *Cauchy Principal Value*:

The Cauchy Principal Value

We determine the value of an improper integral by deleting a symmetric neighborhood around the singularity.

Example

Consider the integral

$$\int_{-a}^{b} \frac{1}{t} dt, \qquad a, b > 0$$

Delete now the region $(-\epsilon, \epsilon)$ around zero and take the limit, letting $\epsilon \to 0$:

$$\lim_{\epsilon \to 0} \left(\int_{-a}^{-\epsilon} \frac{1}{t} dt + \int_{\epsilon}^{b} \frac{1}{t} \right) = \lim_{\epsilon \to 0} (log(\epsilon/a) + log(b/\epsilon)) = log(b/a)$$

Remark

Note that it is crucial for the limit to exist that we delete a symmetric region (-ϵ, ϵ)

Definition

We define the *Cauchy Principal Value* of an Integral with a singularity in a point y to be:

$$P.V. \int_\Omega f(x)dx := \lim_{\epsilon \to 0} \int_{\Omega \setminus |y-x| \leq \epsilon} f(x)dx$$

- For integrals with a strong singularity: interpret it as a CPV-integral.
- But: For hypersingular integrals this limit might not exist.

Example

Strongly singular integrals where we can apply the CPV approach:

$$\text{in 2-D: } \frac{1}{r} \qquad \text{in 3-D: } \frac{1}{r^2}$$

Hypersingular integrals and Hadamard finite part

Example

Consider the 1-dimensional integral:

$$\int_a^b \frac{1}{t^2} dt$$

CPV does not give us a limit. However, we can take the finite part of the CPV:

$$CPV \int_a^b \frac{1}{t^2} dt = \lim_{\epsilon \to 0} \int_a^{-\epsilon} \frac{1}{t^2} + \int_\epsilon^b \frac{1}{t^2} = \lim_{\epsilon \to 0} \frac{2}{\epsilon} + (\frac{1}{a} - \frac{1}{b}).$$

Whereas the first term diverges, the second term is finite. This is the *Hadamard Finite part*.

The Laplace equation

Consider the homogeneous Laplace equation

$$\begin{aligned} \Delta u(x) &= 0 \qquad \text{on } \Omega \subset \mathbb{R}^d \\ \gamma_D(u) &= g \end{aligned}$$

Recall the integral representation formula :

$$\gamma_D u(x) = \gamma_D \int_\Gamma \gamma_N G(x,y) \gamma_D u(y) ds_y - \gamma_D \int_\Gamma \gamma_{N,y} u(y) G(x,y) ds_y,$$

or in terms of Boundary Integral Operators:

$$\gamma_D u = (\frac{1}{2} I - K_0) \gamma_D u + V_0 \gamma_N u$$

$$\rightarrow (-\frac{1}{2} I + K_0) \gamma_D u = V_0 \gamma_N u$$

Aim:

Solving this problem, i.e. find complete Cauchy data.

Two possible approaches

1. Collocation Method$\rightarrow$ *problematic*!
2. Variational method (Galerkin approach)

Collocation methods

- Ansatz: Enforce the boundary integral equation to hold at a specified number of points: $(x_1, \ldots, x_N)$ and obtain the following system of equations:

$$\begin{aligned} -(\tfrac{1}{2}I - K_0)\gamma_D u(x_0) &= V_0 \gamma_N u(x_0) \\ \vdots \qquad\qquad & \qquad\quad \vdots \\ -(\tfrac{1}{2}I - K_0)\gamma_D u(x_N) &= V_0 \gamma_N u(x_N) \end{aligned}$$

  after taking an approximation ansatz one gets Matrix equations:

$$\mathcal{H}(\gamma_D u(x_i))_i = \mathcal{V}(\gamma_N u(x_i))_i$$

  for $(N+1 \times N+1)$ matrices $\mathcal{H}$ and $\mathcal{V}$, whose coefficients are to be determined.

How to compute the Matrix coefficients?

Take ansatz:

$$\gamma_D u \approx \sum_j \gamma_D u(x_j)\phi_j,$$

- $\bigcup_l \tau_l = \Gamma$ is a discretization of the boundary
- (ϕ_j) basis of $\mathcal{S}_h^p(\Gamma)$.

E.g. for $\mathcal{H}$: in order to get the matrix coefficients, we have to compute:

$$\begin{aligned}\int_\Gamma \gamma_D u(y)\gamma_{N,y} G(x_i,y)ds_y &= \sum_l \int_{\tau_l} \gamma_D u(y)\gamma_{N,y} G(x_i,y)ds_y \\ &\approx \sum_l \sum_j \gamma_D u(x_j) \int_{\tau_l} \phi_j(y)\gamma_{N,y} G(x_i,y)ds_y \end{aligned} \tag{1}$$

So the problem of the matrix elements' computation comes down to approximating the integrals:

$$\int_{\tau_l} \phi_j(y)\gamma_{N,y} G(x_i,y)ds_y$$

Problems occur!

What is problematic about this approach? Consider again the integral:

$$\int_{\tau_l} \phi_j(y)\gamma_{N,y} G(x_i, y) ds_y$$

- resulting matrices are dense
- moreover, there is no underlying structure in the matrix which would make computation and storage easier (like symmetricity).
- Problematic when approximating Hypersingular Integral: to compute $\gamma_{N,x} \int_{\tau_l} \phi_j \gamma_{N,y} G(x_i, y) ds_y$ demands smoothness of the interpolation of γ_D at interpolation points. ($\mathcal{C}^1$ interpolation would be possible but very costly.)
- A better approach: The Variational Method

Computing the Matrix coefficients using a variational method

Program:

1. Recalling the variational approach
2. 2D Laplace problem: Explicit computations
3. 3D Laplace problem: Computing the matrix coefficients numerically
    - Gauss Quadrature
    - A Semianalytic Method

Recall from last time: Boundary integral equations I

Instead of demanding that the Boundary Integral equations be satisfied pointwise, we ask for *weak solutions*: Let

$$K(w) = f, \tag{2}$$

where K is some Boundary Integral operator: $K : H^s(\Gamma) \to H^{s'}(\Gamma)$, where $s, s' \in \{-\frac{1}{2}, \frac{1}{2}\}$. Then we ask:

$$\langle f, \psi \rangle = \langle K(w), \psi \rangle, \tag{3}$$

for any $\psi \in H^{-s'}$. Choose Galerkin bases $\{\phi_j\}$, $\{\psi_i\}$ of $H^s, H^{-s'}$ resp., and discretize w as:

$$w(x) = \sum_j w_j \phi_j(x) \tag{4}$$

We obtain a Matrix equation:

$$\mathcal{K}[w] = [f] \tag{5}$$

Recall from last time: Boundary integral equations II

where w and f are given by:

$$w = [w_1, \ldots, w_N] \tag{6}$$

$$f_j = \langle f, \psi_j \rangle \tag{7}$$

and

$$\mathcal{K}_{ij} = \langle K(\phi_i), \psi_j \rangle. \tag{8}$$

Our main concern: Computation of the coefficients $\mathcal{K}_{ij}$

The homogeneous Laplace equation:The 2-D case

As an example, consider:

$$\Delta u = 0, \qquad \text{on } \Omega \subset \mathbb{R}^2 \tag{9}$$

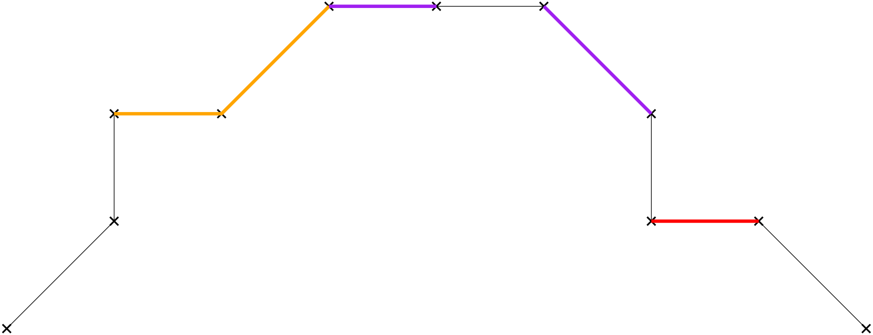
With the Fundamental solution:

$$G(x, y) = -\frac{1}{2\pi} log(|x - y|) \tag{10}$$

discriminate three different cases for the choice of τ_i, τ_j:

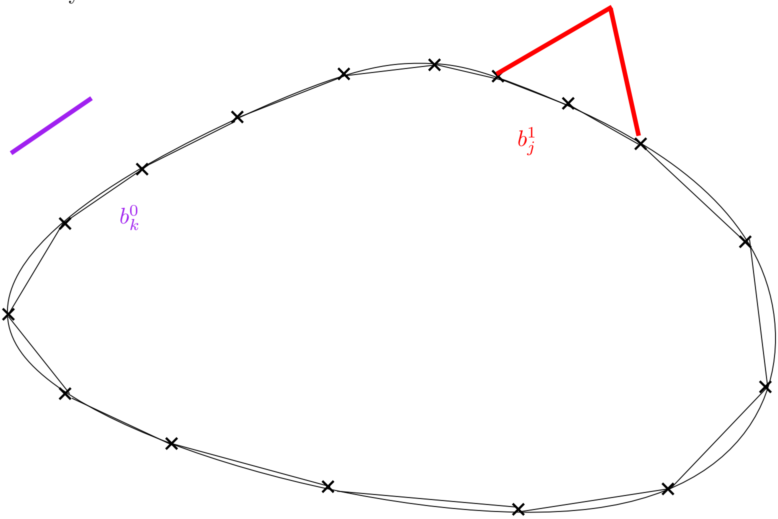
3 different cases

3 different cases for 1-D boundary



Discretized boundary

Discretized boundary Γ in 2D with hat functions



The case of identical panels: Computing $\langle V_0(b_0^0), b_0^0\rangle$

Remark:Use $\mathcal{S}_h^0(\Gamma)$ for $H^{-\frac{1}{2}}(\Gamma)$ (cf. last time) and $\mathcal{S}_h^1(\Gamma)$ for $H^{\frac{1}{2}}$. Recall that $V_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$. Consider the reference element. Let b_0^0 be the corresponding basis function $\in \mathcal{S}_h^0$ over that element. We have:

$$
\begin{aligned}
\langle V_0(b_0^0), b_0^0\rangle &= -\frac{1}{2\pi}\int_0^1\int_0^1 log|x-y|dxdy \qquad (11)\\
&= -\frac{1}{2\pi}\int_0^1\int_{-y}^{1-y} log(|z|)dzdy\\
&= -\frac{1}{2\pi}\int_0^1 (zlog|z|-z)\Big|_{-y}^{1-y} dy\\
&= -\frac{1}{2\pi}\int_0^1 ((1-y)log(1-y)+ylog(y)-1+2y\ )\,dy\\
&= -\frac{1}{\pi}\int_0^1 ulog(u)du
\end{aligned}
\qquad (12)
$$

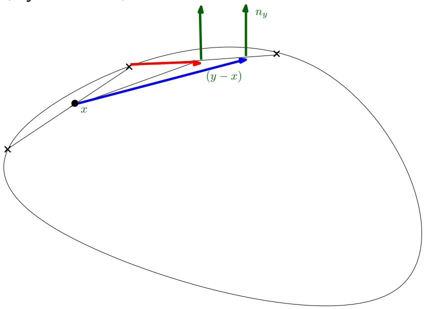
$$
\begin{aligned}
&= -\frac{1}{\pi}\left(\left[u^2\left(\frac{log(u)}{2}-\frac{1}{2u}\right)\right]\Big|_0^1\right)\\
&= \frac{1}{4\pi}
\end{aligned}
$$

Computing $\langle K_0(b_0^1), b_0^0\rangle$

Recall that $K_0 : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ First:

$$\frac{\partial}{\partial n_y} G(x,y) = -\frac{1}{2\pi}\frac{\langle n_y, (y-x)\rangle}{|x-y|^2}$$

Remark: We have that $\langle n_y, y-x\rangle = $ const. always and in case of identical



panels, $\langle n_y, y-x\rangle = 0$

Case of two adjacent panels, not on one line. Take $b_0^1(y) = y$ as a pw. linear basis function on the reference element. Thus

$$\begin{aligned}
\int_{\tau_j}\int_{\tau_i} K_0 b_0^1 &= -\frac{1}{2\pi}\left(\int_{\tau_j}\int_{\tau_i}\frac{b_0^1}{|y-x|^2}ds_y ds_x\right)\\
&= -\frac{1}{2\pi}\left(\int_c^d\int_0^1\frac{s}{(s-t)^2+a^2}dsdt\right)\\
&= -\frac{1}{2\pi}\left(\int_c^d\int_0^1\frac{1}{a}\arctan(\frac{s-t}{a})dsdt + \int_c^d\frac{1}{a}\arctan(\frac{(1-t)}{a})dt\right.\\
&= -\frac{1}{2\pi}\int_c^d \arctan(\frac{(1-t)}{a})(1-t) - \arctan(-t/a)(-t)\\
&- \frac{1}{2\pi}\int_c^d\frac{1}{2}log(1+\frac{(1-t)^2}{a}) + \frac{1}{2}log(1+(t/a)^2)dt
\end{aligned} \tag{13}$$

The last integral is nonsingular,can be evaluated. Similarly, in case of non-adjacent panels, the integral exists.
Want to compute $\langle(\frac{1}{2}I - K_0)\phi, \theta\rangle$. Already seen: $\langle K_0\phi, \theta\rangle$. Now $\langle\phi, \theta\rangle$.
Take $b_0^1(y) = y \in \mathcal{S}_h^1$ and integrate against pw. constant b_0^0 on the reference element:

$$\begin{aligned}\langle b_0^1, b_0^0\rangle &= -\int_0^1 \int_0^1 b_0^1(y)dydx \\ &= \int_0^1 \int_0^1 ydydx \\ &= \int_0^1 (\frac{1}{2}y^2)\Big|_0^1 \\ &= \frac{1}{2}.\end{aligned} \tag{14}$$

(For non-identical elements it is easy to show that the computation still works fine.)

Computing $\langle W_0 b_0^1, b_0^1 \rangle$ I

- Consider the identical case:$\tau_i = \tau_j = (0,1)$ Recall $W_0 : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$.
- Use

$$\gamma_{N,x}\gamma_{N,y} G(x,y) = -\frac{1}{2\pi}\left[-\frac{\langle n_x, n_y\rangle}{r^2} + 2\frac{\langle n_y, (x-y)\rangle\langle n_x, (x-y)\rangle}{r^4}\right]$$

Want to compute $\int_0^1 x\gamma_{N,x}\int_0^1 \gamma_{N,y} G(x,y) y ds_y ds_x$. A different representation of W_0 is:

Computing $\langle W_0 b_0^1, b_0^1 \rangle$ II

$W_0(\phi) = -\int_0^1 \gamma_{N,x}\gamma_{N,y} G(x,y)(\phi(y)-\phi(x))ds_y$. Thus it is enough to compute:

$$
\begin{aligned}
&= -\frac{1}{2\pi}\int_0^1 x \int_0^1 y \cdot \left(-\frac{\langle n_x, n_y\rangle}{r^2} + 2\frac{\langle n_y,(x-y)\rangle\langle n_x,(x-y)\rangle}{r^4}\right) ds_x ds_y \\
&= -\frac{1}{2\pi}\int_0^1 t \int_0^1 s \left(-\frac{1}{(s-t)^2} + 2\frac{(n_y\cdot(x-y))(n_x\cdot(x-y))}{r^4}\right) dsdt \\
&= \frac{1}{2\pi}\int_0^1 t \int_0^1 \frac{s}{(t-s)^2} dsdt
\end{aligned}
\tag{15}
$$

- Singularity is of order $(s-t)^2$, the integral does not have a finite limit. It can be shown that a term $log(\epsilon^2)$ remains when applying the CPV operator, thus Hadamard finite part has to be taken. Analytically, however, the $log(\epsilon^2)$ term cancels with the integration over a corresponding adjacent element.

Computing $\langle W_0 b_0^1, b_0^1 \rangle$ III

Remark

collocation in general fails to capture this!

For the Hypersingular operator, we really benefit from using Galerkin methods rather than Collocation.

Definition (Rotation of a scalar function, 2-D case)

Let v be a scalar function on Γ.

$$\overrightarrow{curl} v = (\frac{\partial}{\partial x_2}\tilde{v}(x), \ -\frac{\partial}{\partial x_1}\tilde{v}(x))^T, \tag{16}$$

where $\tilde{v}$ defines an extension of v into a small neighborhood $\subset \mathbb{R}^3$ of Γ

We introduce:

$$\begin{aligned} curl_\Gamma v(x) &:= n \cdot \overrightarrow{curl} \tilde{v} \\ &= n_1(x)\frac{\partial}{\partial x_2}\tilde{v}(x) - n_2(x)\frac{\partial}{\partial x_1}\tilde{v}(x)) \end{aligned} \tag{17}$$

It then can be shown that for $u, v \in H^{\frac{1}{2}}$ it holds:

$$\langle W_0(u), v\rangle_\Gamma = -\frac{1}{2\pi}\int_\Gamma curl_\Gamma v(x) \int_\Gamma log|x-y| curl_\Gamma u(y) ds_y ds_x \tag{18}$$

Rewriting W_0 for 3D

Similarly in the 3-D case one obtains:

$$\langle W_0(u), v\rangle_\Gamma = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{curl_\Gamma u(y) curl_\Gamma v(x)}{|x-y|} ds_x ds_y \tag{19}$$

Here we used:

$$curl v = \nabla \times v(x) \tag{20}$$

and

$$curl_\Gamma u = n(x) \times \nabla\tilde{u} \qquad x \in \Gamma \tag{21}$$

where $\tilde{u}$ again is an extension of u into a small neighborhood $\subset \mathbb{R}^3$ of $x \in \Gamma$.)

- curl of $\phi \in \mathcal{S}_h^1$ is constant on each triangle, by linearity of functions in $\mathcal{S}_h^1$.
- moreover n is constant on each triangle.
- therefore: $curl_\Gamma u(y) curl_\Gamma v(x)$ can be taken out of the integral $\rightarrow$ reduces to case: $\langle V_0 b_i^0, b_j^0\rangle$ times some constant.

By using a Variational method, we reduce the (hard) computation of the hypersingular integral to the weakly singular!!!

Quadrature I

In 3D: Numerical evaluation of integrals: Let Ω be a domain in $\mathbb{R}^d$, d=2,3. Let $\Gamma = \partial\Omega$. Recall that $\forall\ \tau \in \mathcal{G}$ ($\mathcal{G}$ a triangulation of Γ)there is a parametrization

$$\chi_\tau : \hat{\tau} \to \tau, \tag{22}$$

where $\hat{\tau}$ denotes the reference element.
Write an integral of a function $v(x) : \tau \to \mathbb{R}$ over τ as an integral $\hat{\tau}$, using the rule of transformation known from calculus, thus:

$$\int_\tau v(x)dx = \int_{\hat{\tau}} v|_\tau \circ \chi_\tau(\hat{x}) g_\tau(\hat{x}) d\hat{x}, \tag{23}$$

where g denotes the Jacobian determinant of χ_τ.

Quadrature II

Definition (Numerical Quadrature on the reference element)

A *Numerical Quadrature on the reference element* is a map:

$$Q :\to C^0(\bar{\hat{\tau}}) \to \mathbb{R} \tag{24}$$

$$Q(v) = \sum_{i=1}^{n} w_{i,n} v(\xi_{i,n}) \tag{25}$$

The $w_{i,n}$ are called *weights*, the $\xi_{i,n}$ Quadrature points.

Definition (Quadrature Error)

The numerical *Quadrature Error* is given by:

$$E_{\hat{\tau}} = \int_{\hat{\tau}} v(x)dx - Q(v) \tag{26}$$

Quadrature III

Definition (Exactness)

A numerical quadrature is said to be *exact of degree m*, $m \in \mathbb{N}$ if $E_\tau = 0$, $\forall v \in \mathbb{P}_m$, where $\mathbb{P}_m$ denotes the space of polynomials of degree m.

Example (A very simple one)

Consider the reference triangle with nodes (0,0), (1,0), (1,1). Then

$$Q(v) = \frac{v(2/3, 1/3)}{2} \tag{27}$$

has degree 1.

Proof.

Let p(x)=ax+by+c be a polynomial in (x,y) of degree 1. Integrating over the triangle yields:

$$\int_0^1 \int_0^x ax + by + c \; dydx = \int_0^1 ax^2 + \frac{1}{2}bx^2 + cx \; dx \tag{28}$$

$$= \frac{a}{3} + \frac{b}{6} + \frac{c}{2}, \tag{29}$$

whereas

$$Q(p) = \frac{v(2/3, 1/3)}{2} \tag{30}$$

$$= a \cdot \frac{2}{3} + b \cdot \frac{1}{3} + c, \tag{31}$$

on the other hand, it easily can be shown that equation does no longer hold for polynomials of degree ≥ 2. So indeed order of exactness is 1 $\square$

Doing Quadrature on the triangular elements

Recall that for an integral

$$I(f) = \int_{[0,1]^4} f(x)dx \tag{32}$$

we have a the Gaussian Quadrature of order $n = (n_1, n_2, n_3, n_4)$ given by:

$$Q[f] = \sum_{i}^{n_1} \sum_{j}^{n_2} \sum_{k}^{n_3} \sum_{l}^{n_4} \omega_{i,n_1} \omega_{j,n_2} \omega_{k,n_3} \omega_{l,n_4} f(x_{i,n_1}, x_{j,n_2}, x_{k,n_3}, x_{l,n_4}) \tag{33}$$

where Gauss points and weights are given. This rule yields the correct result for polynomials of degree (2n-1) where n+1 Gauss quadrature points are used.

Example:Gauss Legendre Quadrature over Triangles

Consider a triangle with vertices P_1, P_2, P_3.

Definition (Barycentric coordinates)

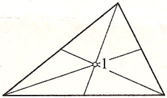
Write a point x inside the triangle as:

$$x = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3.$$

$(\alpha_1, \alpha_2, \alpha_3)$ are called the *barycentric coordinates* of x.

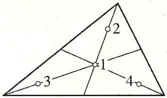
We use these coordinates to represent Gauss Points within the triangle.

A simple 1-point rule. [1]

Figure	m	p	i	L^e_{1i}	L^e_{2i}	L^e_{3i}	W_i
	1	1	1	0.33333333	0.33333333	0.33333333	1.00000000

[1] Source:"'The Finite Element Method in Electromagnetics"', Jianming Jin, 2nd Edition, John Wiley and Sons, Inc.,New York, USA

A three-point rule [2]

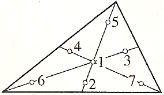


4	3	1	0.33333333	0.33333333	0.33333333	−0.56250000
		2	0.73333333	0.13333333	0.13333333	0.52083333
		3	0.13333333	0.73333333	0.13333333	0.52083333
		4	0.13333333	0.13333333	0.73333333	0.52083333

(continued)

[2] Source:"'The Finite Element Method in Electromagnetics"', Jianming Jin, 2nd Edition, John Wiley and Sons, Inc.,New York, USA

A 7-point rule [3]



7	4	1	0.33333333	0.33333333	0.33333333	0.22500000
		2	0.05961587	0.47014206	0.47014206	0.13239415
		3	0.47014206	0.05961587	0.47014206	0.13239415
		4	0.47014206	0.47014206	0.05961587	0.13239415
		5	0.79742699	0.10128651	0.10128651	0.12593918
		6	0.10128651	0.79742699	0.10128651	0.12593918
		7	0.10128651	0.10128651	0.79742699	0.12593918

[3] Source:"'The Finite Element Method in Electromagnetics"', Jianming Jin, 2nd Edition, John Wiley and Sons, Inc.,New York, USA

An estimate of error (without proof)

Theorem

Let $\bigcup_l \tau_l$ be a triangulation of Γ into plane triangles. Suppose that the ratio $\frac{diam\ \tau}{diam\ r}$ for elements τ, r, $\tau \neq r$ is bounded. Let m be the order of exactness of the Quadrature method used.
then we have $\forall v \in H^{\tilde{m}}(\tau)$, with $\tilde{m} = max\{2, m+1\}$:

$$|E_{\tau(v)}| \leq C h_\tau^{m+2} \|v\|_{H^{\tilde{m}}(\tau)}, \tag{34}$$

for some constant C and $h_\tau = diam(\tau)$.

Quadrature method: Pro and Contra

Pro:

- Well-suited for implementation
- High accuracy can be achieved

Contra:

- computationally costly: e.g., in 3D case: Consider the integration of V_0. for one double integral the cost is $\mathcal{O}(n_{QP}^4)$ (n_{QP}=Number of Quadrature points). If $dim(\mathcal{S}_h^0) = M$, we have M^2 entries to compute. The total cost for this matrix is therefore $\mathcal{O}(n_{QP}^4 \cdot M^2)$
- near singularities: the values get very large. Refinement of triangulation $\rightarrow$ values become larger as quadrature points lie even closer. Also: refining the mesh $\rightarrow$ much more cost

- First approach deals with the second problem: Singular Integrals.
- Recall that our problem is the computation of integrals of the form:

$$\int_{\tau_j} \int_{\tau_i} \frac{f(x,y)}{\|x-y\|} dydx$$

First, we focus on the inner integral. Consider $h(x,y)$ that has a first order singularity at the point $(0,0)$ over the reference triangle $\hat{\tau}$ with vertices $(0,0),(1,0),(1,1)$.

$$\int_{\tau_j} h(x,y)dS = \int_0^1 \int_0^x h(x,y)dydx \qquad (35)$$

Using Taylor expansion, it follows that h can be written as $\frac{g(x,y)}{\sqrt{x^2+y^2}}$, where g is an analytic function over Ω. Perform change of variables:

$$y = xu$$

The above integral then equals to:

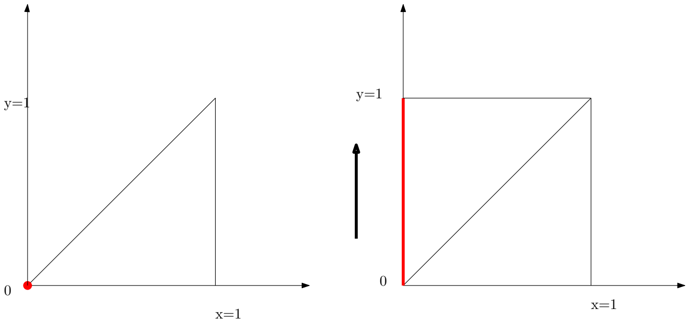
$$\int_0^1 \int_0^1 \frac{g(x, xu)}{\sqrt{x^2 + (xu)^2}} x \; dudx$$

Now we see that x cancels in this expression and we get:

$$\int_0^1 \int_0^1 \frac{g(x, xu)}{\sqrt{1 + u^2}} dudx$$

Clearly, this function has no singularity.

Duffy's Trick



- After this transformation, we can use a Quadrature for the double integral and no longer have the problem of singular integration.
- But: We have only dealt with the 2nd problem i.e. singularity. The problem of high cost of the double quadrature remains!
- Semianalytic method!

Another method: Semianalytic Method for 3-D

We consider another method for the 3-D Laplace problem. Recall that the fundamental solution is given by:

$$G(x, y) = \frac{1}{4\pi \|x - y\|} \tag{36}$$

We now want to compute the single layer potential boundary operator, i.e.

$$\langle V_0(b_i^0), b_j^0 \rangle = \int_{\tau_j} \int_{\tau_i} \frac{1}{4\pi \|x - y\|} ds_y ds_x \tag{37}$$

Idea : compute inner integral analytically, then solve the outer integral using Quadrature. $\rightarrow$ only one quadrature to do!

Aim:

$$solve \int_{\tau_i} \frac{1}{4\pi \|x-y\|} ds_y ds_x .$$

Remark that all our further computations are for the 3-D case! We have:

$$\Delta\|x-y\| = \frac{2}{\|x-y\|} \tag{38}$$

$\rightarrow$

$$\int_{\tau_i} \frac{1}{\|x-y\|} ds_y = \frac{1}{2} \int_{\tau_j} \Delta\|x-y\| ds_y \tag{39}$$

Lemma

Decomposition of the Laplace Operator

$$\Delta u = \frac{\partial^2}{\partial n^2} u + 2H_n \frac{\partial u}{\partial n} + \Delta_\Gamma u \tag{40}$$

where

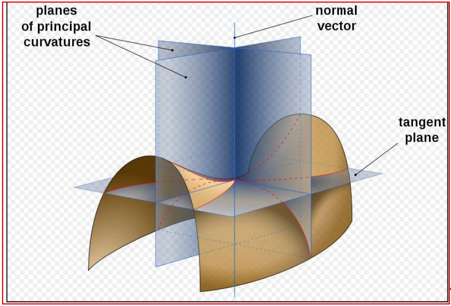
$$\Delta_\Gamma u = -curl_\Gamma \overrightarrow{curl_\Gamma} u \tag{41}$$

where H_n *denotes the mean curvature of* Γ. *The operator* $\overrightarrow{curl_\Gamma}$ *is called tangential rotation, input is a scalar function, and defined as:* $\overrightarrow{curl_\Gamma}(u) := curl(\tilde{u}n)|_\Gamma$. *the scalar function* $curl_\Gamma(u) := (curl \tilde{u} \cdot n)|_\Gamma$ *is called surfacic rotation (remark that its input is a vector).*

The mean curvature/principle curvatures

Definition (Principal curvatures)

Consider the curvature operator $\mathcal{R} = \nabla n$, which acts on the tangent plane of a surface. Its two eigenvalues κ_1, κ_2 are called principal curvatures. The mean curvature is defined as: $H_n := \frac{1}{2}(\kappa_1 + \kappa_2)$.



[4] Remark: On flat triangles: $H_n = 0$

[4]Source: Wikipedia

We put the decomposed form of the Laplace operator into equation (39) and obtain:

$$\begin{aligned} F(x) :&= \int_{\tau_i} \frac{1}{\|x-y\|} ds_y \\ &= \frac{1}{2} \int_{\tau_j} \Delta \|x-y\| ds_y \\ &= \frac{1}{2} \int_{\tau_j} \frac{\partial^2}{\partial n^2} \|x-y\| + curl_\Gamma \overrightarrow{curl}_\Gamma \|x-y\| ds_y \end{aligned} \tag{42}$$

Computation of the inner integral

Using the definition of $\overrightarrow{curl_\Gamma}(u)$ and of the *curl* this is equivalent to:

$$\frac{1}{2}\int_{\tau_j}\frac{\partial^2}{\partial n^2}\|x-y\| - \frac{1}{2}\int_{\tau_j} n\cdot\nabla\times\nabla(\|x-y\|n)ds_y \tag{43}$$

We consider the two integrals obtained seperately. First, remarking that:

$$\nabla\|x-y\| = \frac{(y-x)}{\|x-y\|}$$

This gives us:

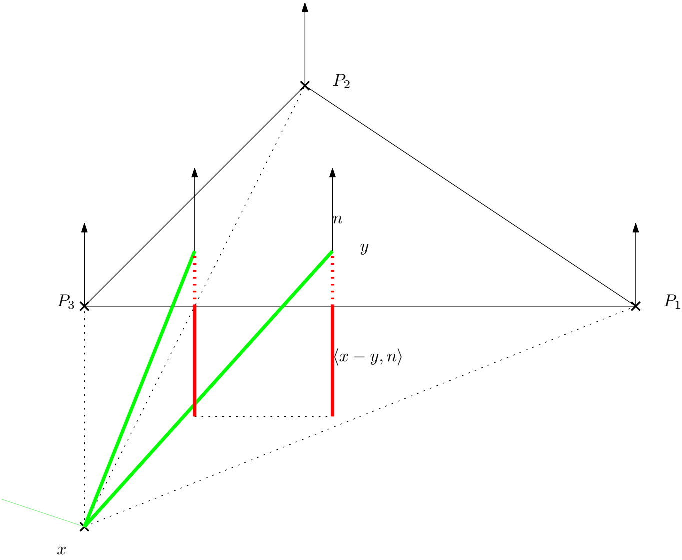
$$\begin{aligned}\frac{\partial^2}{\partial n^2}\|x-y\| &= \frac{\partial}{\partial n}(\nabla\|x-y\|\cdot n)\\ &= \frac{\partial}{\partial n}\left(\frac{(y-x)}{\|x-y\|}\cdot n\right)\end{aligned} \tag{44}$$

Computation of the inner integral

One can find that

$$(y - x) \cdot n =: C(x, \tau_j)$$

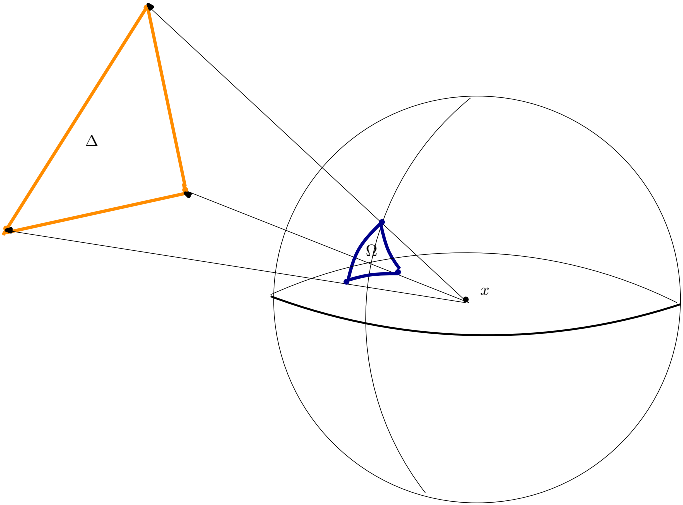
is just the length of the projection onto *span*(n) of the distance between x and the triangle τ_j, and thus constant for fixed x. (See picture)


P_2
n
y
P_3
P_1
$\langle x - y, n \rangle$
x

Now move $C(x,\tau_j)$ outside the integral. The first term of the sum (43) reduces to:

$$
\begin{aligned}
\frac{1}{2}C(x,\tau_j)\int_{\tau_j}\frac{\partial}{\partial n}\left(\frac{1}{\|x-y\|}\right)ds_y \\
&= \frac{1}{2}C(x,\tau_j)\int_{\tau_j}\nabla\left(\frac{1}{\|x-y\|}\right)\cdot n\ ds_y \\
&= -\frac{1}{2}C(x,\tau_j)\int_{\tau_j}\frac{(y-x)}{\|x-y\|^3}\cdot n\ ds_y \quad (45)
\end{aligned}
$$

This integral now can be computed efficiently, as it describes a geometric relationship between x and the triangle τ_j, the *solid angle*, for whose computation efficient algorithms exist.


Δ
Ω
x

Computation of the inner integral: 2nd term I

We now want to compute the second term, i.e.

$$\frac{1}{2}\int_{\tau_j} n \cdot \nabla \times \nabla \times (\|x-y\| \cdot n) ds_y$$

Recall Stokes' Theorem:

Theorem (Stokes' Thm)

Let $\Omega \subset U \subset \mathbb{R}^3$ be a regular surface and U an open subset of $\mathbb{R}^3$.Let F be a vector valued function on U.Then we have:

$$\int_{\Omega} curlF \cdot n \; dS = \oint_{\partial\Omega} F \cdot dr \tag{46}$$

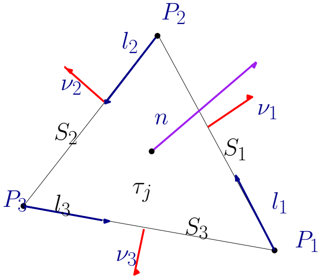
Computation of the inner integral: 2nd term II

We put this result into the previous equation and get:

$$\frac{1}{2}\int_{\tau_j} n \cdot \nabla \times \nabla \times (\|x-y\| \cdot n) ds_y = \oint_{\partial\tau_j} \nabla \times (\|x-y\| n) d\vec{r}$$
$$= \oint_{\partial\tau_j} \nabla(\|x-y\|) \times n d\vec{r}$$
$$= \oint_{\partial\tau_j} \frac{(y-x)}{\|x-y\|} \times n d\vec{r}, \tag{47}$$

where we used that the field of normals is a gradient, and therefore $\nabla \times n = 0$.

It remains to compute the contour integral along the boundary of τ_j. We can write this integral as a sum of three integrals, one along each edge of the triangle τ_j.



Compare with the notations on the picture to convince yourself that:

$$\oint_{\partial\tau_j} \frac{(y-x)}{\|x-y\|} \times n \, d\vec{r}$$

$$= \sum_{k=1}^{3} \int_{S_k} \frac{(y-x)}{\|x-y\|} \times n \cdot l_k dS_k$$

$$= -\sum_{k=1}^{3} \int_{S_k} \frac{(y-x)}{\|x-y\|} \cdot \nu_k dS_k \tag{48}$$

, where we also used that $(a \times b) \cdot c = -(a \times c) \cdot b$. As in the first computation we have that:

$$(y-x) \cdot \nu_k = (P_k - x) \cdot \nu_k$$

is constant and therefore can again be taken out of the integral. So the above expression becomes:

$$-\frac{1}{2}\sum_{k=1}^{3}(P_k - x)\nu_k \int_{S_k} \frac{1}{\|x-y\|} dS_k \tag{49}$$

This integral then can be computed analytically without much trouble.

Summary

We wanted to compute singular integral contributions in order to solve the Laplace problem with Dirichlet BC using Boundary element methods. We have seen the following:

- Before doing so, we explained why we choose Galerkin methods over collocation. Of particular importance: Computation of Hypersingular integral
- For 2-D, integrals can be explicitly computed
- For 3-D, we generally have to use some numerical method:
- We saw: Quadrature may lead to accurate results, however not near singularities.
- Another issue: High cost of quadrature methods!
- For singular integrals, transformation methods exists that make the integral nonsingular (Duffy trick)
- To deal with the problem of high-costs in double quadratures, we encoutered an altoghether different method, using a seminanalytic approach .