BEM: approximation bases and convergence analyses

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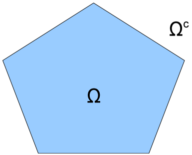
November 4, 2010

The Dirichlet BVP

Let $\Omega \subset \mathbb{R}^3$ be a bounded Polyhedron.

The Dirichlet problem for the Laplace equation in Ω^c is

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^c, \\ u = g_D & \text{on } \Gamma, \\ |u(x)| = \mathcal{O}(\|x\|^{-1}) & \text{for } \|x\| \to \infty. \end{cases}$$



For every $g_D \in H^{1/2}(\Gamma)$ there is exactly a solution $u \in H^1_{loc}(\Omega^c)$ to the variational formulation of the DBVP.

We use the indirect approach, which means that we make the Ansatz

$$u(x) = (\Psi^0_{SL}\varphi)(x) = \int_\Gamma \frac{\varphi(y)}{4\pi\|x-y\|} ds_y, \quad x \in \Omega^c.$$

The density function φ is the solution of the boundary integral equation

$$V_0\varphi = g_D \quad \text{on } \Gamma. \tag{1}$$

Recall that the weakly singular boundary integral operator

$$V_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is defined through

$$(V_0\phi)(x) := \gamma_D(\Psi_{SL}^0\phi)(x) \quad \text{for } x \in \Gamma.$$

Moreover it is bounded and elliptic, and thus invertible.

The Galerkin-BEM is founded on the variational formulation of (1):

find $\varphi \in H^{-1/2}(\Gamma)$ such that

$$\langle V_0 \varphi, \tau \rangle_\Gamma = \langle g_D, \tau \rangle_\Gamma$$

for all $\tau \in H^{-1/2}(\Gamma)$.

The direct approach would be

$$u(x) = \int_\Gamma G_0(x,y)\gamma_N u(y) ds_y - \int_\Gamma \gamma_N G_0(x,y) g(y) ds_y, \quad x \in \Omega^c$$

where $\gamma_N u$ is the unique solution of

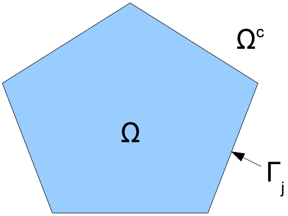
$$\langle V_0 \gamma_N u, \tau \rangle_\Gamma = \langle (\frac{1}{2} I + K_0) g_D, \tau \rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

Galerkin approximation

The idea of Galerkin approximation is to consider the variational problem on a much smaller (finite) subspace and to try to solve the integral equation there.

Let assume that the boundary $\Gamma := \partial\Omega^{-}$ is the union of finite disjoint sides Γ_j:

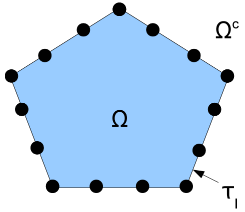
$$\overline{\Gamma} = \bigcup_{j=1}^{J} \overline{\Gamma_j}.$$



Now consider a sequence $\{\Gamma_N\}_N \in \mathbb{N}$ of meshes

$$\Gamma_N = \bigcup_{l=1}^{N} \overline{\tau}_l$$

with boundary elements τ_l.



Moreover we assume that for any l there's a unique index j with $\tau_l \subset \Gamma_j$.

The local mesh size of τ_l is

$$h_l := \sup_{x,y \in \tau_l} \|x - y\|.$$

Remark: in this presentation $\tau_l' s$ are chosen to be triangles.

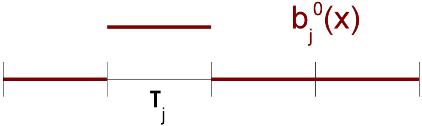
$\mathcal{S}_h^0(\Gamma)$ is the space of piecewise constant functions.

$$\mathcal{S}_h^0(\Gamma) := span\{b_k^0\}_{k=1}^M$$

where

$$b_k^0(x) = \begin{cases} 1 & \text{for } x \in \tau_k, \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly we can define the space of piecewise polynomial functions $\mathcal{S}_h^p(\Gamma)$.



Since

$$\mathcal{S}_h^p(\Gamma) \subset H^{-1/2}(\Gamma)$$

we can limit the variational problem above to $\mathcal{S}_h^p(\Gamma)$.

To do this, we substitute φ with

$$\varphi_h(x) := \sum_{k=1}^{M} \varphi_k \cdot b_k^p(x).$$

Let $p \in \mathcal{S}_h^p(\Gamma)$, then

$$
\begin{aligned}
\|p\|_{H^{-1/2}(\Gamma)} &:= \sup_{\phi \in H^{1/2}(\Gamma), \|\phi\|=1} \langle p, \phi \rangle_\Gamma \\
&= \sup_{\phi \in H^{1/2}(\Gamma), \|\phi\|=1} \int_\Gamma p \cdot \phi dS \\
&\leq \sup_{\phi \in H^{1/2}(\Gamma), \|\phi\|=1} \sup_{x \in \Gamma}(p(x)) \int_\Gamma \phi dS \\
&\leq C(p) \sup_{\phi \in H^{1/2}(\Gamma), \|\phi\|=1} \|1\|_{L^2(\Gamma)} \|\phi\|_{L^2(\Gamma)} \\
&\leq C(p) \cdot C(\Gamma).
\end{aligned}
$$

The Galerkin variational formulation of the Dirichlet BVP reads to find $\varphi_h \in \mathcal{S}_h^p(\Gamma)$ such that

$$\langle V_0 \varphi_h, \tau_h \rangle_\Gamma = \langle g_D, \tau_h \rangle_\Gamma \quad \text{for all } \tau_h \in \mathcal{S}_h^p(\Gamma).$$

This problem is equivalent to find φ_h such that

$$\langle V_0 \varphi_h, b_l^p \rangle_\Gamma = \langle g_D, b_l^p \rangle_\Gamma \quad \text{for all } l = 1, \cdots, M.$$

By inserting the definition of φ_h, we have

$$\sum_{k=1}^{M} \varphi_k \langle V_0 b_k^p, b_l^p \rangle_\Gamma = \langle g_D, b_l^p \rangle_\Gamma \quad \text{for all } l = 1, \cdots, M.$$

These M linear equations can be collected in the following linear system

$$\left(\langle V_0 b_k^p, b_l^p \rangle_\Gamma \right)_{l,k=1}^{M} \cdot (\varphi_k)_{k=1}^{M} = \left(\langle g_D, b_l^p \rangle_\Gamma \right)_{l=1}^{M}.$$

We have reduced the original problem to a linear problem.

Remark: we have chosen $\{b_l^p\}_l$ as an orthonormal system, this is possible because φ_h is independent of the basis.

Now the difficulties lies in the computation of the stiffness matrix

$$\left(\langle V_0 b_k^p, b_l^p\rangle_\Gamma\right)_{l,k=1}^{M}$$

and of the load vector

$$\left(\langle g_D, b_l^p\rangle_\Gamma\right)_{l=1}^{M}.$$

Note:

- the stiffness matrix and the load vector are full and usually can't be found analytically,
- the numerical computation of these integral is also extremely difficult (see the following presentation)!
- Moreover it holds $cond_2\left(\langle V_0\varphi_k^0,\varphi_l^0\rangle_\Gamma\right)_{l,k=1}^{M} \leq ch^{-1}$.

For the rest of the presentation we will assume that they have been computed exactly.

The stiffness matrix is

- symmetric (since the kernel is),
- positive definite (also because of the kernel).

Thus the linear problem has exactly a solution φ_h (which is called Galerkin solution).

Moreover the Galerkin solution satisfies:

$$\langle V_0(\varphi - \varphi_h), \eta \rangle_\Gamma = 0 \quad \text{for every } \eta \in \mathcal{S}_h^p$$

and

$$\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)} \leq C \min_{\eta \in \mathcal{S}_h^p} \|\varphi - \eta\|_{H^{-1/2}(\Gamma)}.$$

Galerkin orthogonality: let $\eta \in \mathcal{S}_h^p$, then

$$\begin{aligned} \langle V_0(\varphi - \varphi_h), \eta \rangle_\Gamma &= \langle V_0 \varphi, \eta \rangle_\Gamma - \langle V_0 \varphi_h, \eta \rangle_\Gamma \\ &= \langle g_D, \eta \rangle_\Gamma - \langle g_D, \eta \rangle_\Gamma \\ &= 0. \end{aligned}$$

Quasi-optimal convergence: let $\eta \in \mathcal{S}_h^p$, then

$$\begin{aligned}
\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)} &\leq C\langle V_0(\varphi - \varphi_h), \varphi - \varphi_h\rangle_\Gamma \\
&= C\langle V_0(\varphi - \varphi_h), \varphi\rangle_\Gamma - \langle V_0(\varphi - \varphi_h), \varphi_h\rangle_\Gamma \\
&= C\langle V_0(\varphi - \varphi_h), \varphi\rangle_\Gamma \\
&= C\langle V_0(\varphi - \varphi_h), \varphi\rangle_\Gamma - \langle V_0(\varphi - \varphi_h), \eta\rangle_\Gamma \\
&\leq C\|V_0\|\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)}\|\varphi - \eta\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Lemma: let $\{\Gamma_n\}_{n\in\mathbb{N}}$ be a sequence of meshes with $h^n_{max} \rightarrow 0$. Then the sequence $\{\varphi_{h^n}\}_{n\in\mathbb{N}}$ converges to φ in $H^{-1/2}(\Gamma)$.

In general φ is not continuos, thus we just consider $\varphi_h \in \mathcal{S}_h^0(\Gamma)$

Theorem: let φ be in $H^s(\Gamma)$ for $s \in [0,1]$, then

$$\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)} \leq Ch^{s+1/2}\|\varphi\|_{H^s(\Gamma)}.$$

Let assume that the following lemma is true

Lemma: $\varphi \in H^s(\Gamma)$ for $s \in [0,1]$. Let

$$Q\varphi := \sum_{k=1}^{M} \varphi_k \cdot b_k^0(x)$$

with

$$\varphi_k := \int_{\tau_k} \varphi(x)dx : \int_{\tau_k} 1dx.$$

Then

$$\|\varphi - Q\varphi\|_{L^2(\Gamma)} \leq ch^s \|u\|_{H^s(\Gamma)}.$$

Then we have

$$
\begin{aligned}
\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)} &\leq C \min_{\eta \in \mathcal{S}_h^p} \|\varphi - \eta\|_{H^{-1/2}(\Gamma)} \\
&\leq C \|\varphi - Q\varphi\|_{H^{-1/2}(\Gamma)} \\
&= C \sup_{\phi \in H^{1/2}(\Gamma), \|\phi\|=1} |\langle \varphi - Q\varphi, \phi \rangle_\Gamma| \\
&= C \sup_{\phi \in H^{1/2}(\Gamma), \|\phi\|=1} |\langle \varphi - Q\varphi, \phi - Q\phi \rangle_\Gamma| \\
&\leq C \|\varphi - Q\varphi\|_{L^2(\Gamma)} \sup_{\phi \in H^{1/2}(\Gamma), \|\phi\|=1} \|\varphi - Q\varphi\|_{L^2(\Gamma)} \\
&\leq c h^{s+1/2} \|u\|_{H^s(\Gamma)}.
\end{aligned}
$$

The BEM solution of the BVP

The approximation of the solution of the Dirichlet BVP is then

$$u_h(x) := \int_\Gamma \frac{\varphi_h(y)}{4\pi\|x-y\|} ds_y, \quad x \in \Omega^c.$$

Thus we have the pointwise error estimate

$$|u(x) - u_h(x)| \leq C\|\varphi - \varphi_h\|_{H^{1/2}(\Gamma)}.$$

Theorem: let $u \in H^1(\Omega)$ be the solution of the DBVP, then

$$\|u - u_h\|_{H^1(\Omega)} \leq c\|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)}.$$

Thus, if $\varphi \in H^1(\Gamma)$ we have

$$\|u - u_h\|_{H^1(\Omega)} \leq ch^{3/2}\|\varphi - \varphi_h\|_{H^1(\Gamma)}.$$

First define $\tilde{g} := V_0 \varphi_h$ and $\tilde{u} := \Psi^0_{SL} \varphi_h$.

Then recall that the Inverse Trace Theorem states that the trace operator

$$\gamma_d : H^1(\Omega^c) \to H^{1/2}(\Gamma)$$

has a continuous right inverse operator

$$\mathcal{E} : H^{1/2}(\Gamma) \to H^1(\Omega^c).$$

Thus we can define $u_0 := u - \mathcal{E}g$ and $\tilde{u_0} := \tilde{u} - \mathcal{E}\tilde{g}$ as functions in $H_0^1(\Omega^c)$. Since u and $\tilde{u}$ satisfy

$$\langle \nabla u, \nabla v \rangle_{H_0^1(\Omega)} = 0 \quad \text{for all } v \in H_0^1(\Omega)$$

$$\langle \nabla \tilde{u}, \nabla v \rangle_{H_0^1(\Omega)} = 0 \quad \text{for all } v \in H_0^1(\Omega),$$

we have

$$\langle \nabla (u_0 + \mathcal{E}g), \nabla v \rangle_{H_0^1(\Omega)} = 0 \quad \text{for all } v \in H_0^1(\Omega)$$

$$\langle \nabla (\tilde{u_0} + \mathcal{E}\tilde{g}), \nabla v \rangle_{H_0^1(\Omega)} = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

We subtract the two equations and we obtain

$$\langle \nabla(u_0 - \tilde{u}_0), \nabla v \rangle_{H_0^1(\Omega)} = \langle \nabla(\mathcal{E}(\tilde{g} - g)), \nabla v \rangle_{H_0^1(\Omega)} \quad \text{for all } v \in H_0^1(\Omega).$$

Then

$$\begin{aligned} \|u_0 - \tilde{u}_0\|^2_{H_0^1(\Omega)} &\leq C\langle \nabla(u_0 - \tilde{u}_0), \nabla(u_0 - \tilde{u}_0) \rangle_{H_0^1(\Omega)} \\ &= C\langle \nabla(\mathcal{E}(\tilde{g} - g)), \nabla(u_0 - \tilde{u}_0) \rangle_{H_0^1(\Omega)} \\ &\leq C\|\mathcal{E}(\tilde{g} - g)\|_{H_0^1(\Omega)} \|u_0 - \tilde{u}_0\|_{H_0^1(\Omega)}. \end{aligned}$$

Finally

$$\begin{aligned}
\|u-\tilde{u}\|_{H_0^1(\Omega)} &\leq \|u_0-\tilde{u_0}\|_{H_0^1(\Omega)} + \|\mathcal{E}(\tilde{g}-g)\|_{H_0^1(\Omega)} \\
&\leq C\|\mathcal{E}(\tilde{g}-g)\|_{H_0^1(\Omega)} \\
&\leq C\|\tilde{g}-g\|_{H^{1/2}(\Omega)} \\
&= C\|V_0(\varphi_h-\varphi)\|_{H^{1/2}(\Omega)} \\
&\leq C\|\varphi_h-\varphi\|_{H^{-1/2}(\Omega)}.
\end{aligned}$$

The Dirichlet BVP for the Helmholtz Equation in Ω^c is

$$\begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega^c, \\ u = g_D & \text{on } \Gamma, \\ |u(x)| = \mathcal{O}(\|x\|^{-1}) & \text{for } \|x\| \to \infty \\ |\frac{\partial u}{\partial r} - iku| \leq \mathcal{O}(\|x\|^{-2}) & \text{for } \|x\| \to \infty. \end{cases}$$

We know that the solution $u \in H^1_{loc}(\Omega^c)$ is unique, thus we could try to work in an analogous way to the Laplace problem. We make the Ansatz

$$u(x) = (\Psi_S L^k \varphi) \quad \text{for } x \in \Omega^c$$

where φ satisfies

$$(V_k \varphi)(x) = g_D(x) \quad \text{for } x \in \Gamma.$$

The Helmholtz Equation

Unfortunately the latter boundary integral equation is not unique solvable for those k such that $k^2 =: \lambda$ is an eigenvalue of the interior Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u_\lambda = \lambda u_\lambda & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma, \end{cases}$$

because in this case the single layer potential $V_k : H^{-1/2}(\Gamma) \to H^{1/2}$ is no more injective.