Boundary Element Methods for Wave Scattering.
Third Topic: The Helmholtz Equation

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1 Steklov-Poincaré Operator

Last time we saw the Calderón projection that satisfies the following system of boundary integral equation for a harmonic $u \in H^1(\Omega)$(we consider only the homogeneous case, i.e. $f = 0$)

$$\begin{pmatrix} \gamma_D u \\ \gamma_N u \end{pmatrix} = \begin{pmatrix} (1-\sigma)I - K_0 & V_0 \\ W_0 & \sigma I + K_0' \end{pmatrix} \begin{pmatrix} \gamma_D u \\ \gamma_N u \end{pmatrix}$$

We proved that V_0 is $H^{-\frac{1}{2}}$-elliptic and therefore, by the Lax-Milgram theorem, invertible. Take the first of the two equation and solve for the Neumann data gives

$$\gamma_N u = V_0^{-1}(\sigma I + K_0)\gamma_D u$$

We thus found an operator

$$S_0 := V_0^{-1}(\sigma I + K_0) : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$$

that maps the Dirichlet datum to the Neumann datum. We call S_0 Steklov-Poincaré operator.

Use the second equation to obtain the symmetric representation of the Steklov-Poincaré operator:

$$S_0 = W_0 + (\sigma I + K_0')V_0^{-1}(\sigma I + K_0)$$

In this form, we see that S_0 admits the same ellipticity estimates as W_0:

$$\langle S_0 v, v\rangle_\Gamma \geq \langle W_0 v, v\rangle_\Gamma$$

using the $H^{-\frac{1}{2}}$-ellipticity of V_0. In particular, S_0 is $H_*^{\frac{1}{2}}(\Gamma)$-elliptic.

2 Wave Equation and Helmholtz Equation

2.1 Wave equation reduces to Helmholtz Equation

The wave equation is

$$\frac{\partial^2}{\partial t^2}\Psi - c^2 \Delta \Psi = 0$$

Assume the solution to be time harmonic:

$$\Psi(t, x) = e^{-iwt}u(x)$$

Then u will satisfy

$$-w^2 u - c^2 \Delta u = 0 \text{ or } -\Delta u - k^2 u = 0 \text{ with } k = \frac{w}{c}$$

and is called Helmholtz equation.

2.2 Fundamental solution

We shall always assume $k \in \mathbb{C}^*$ and $0 \leq \arg k < \pi$. To derive the fundamental solution of the Helmholtz equation, we go over to spherical coordinates (remember that the Laplacian commutes with rotation and $(-\Delta_y - k^2)G_k(x-y) = \delta(x-y)$, so $(-\Delta_z - k^2)G_k(z)$ vanishes $r = |z| = |x-y| > 0$):

$$\frac{1}{r^2}\partial_r\left[r^2 \partial_r \tilde{u}(r)\right] - k^2 \tilde{u}(r) = 0 \text{ where } \tilde{u}(r) = \tilde{u}(|z|) = u(z)$$

Put $v(r) = r\tilde{u}(r)$ and v solves

$$v''(r) + k^2 v(r) = 0.$$

We find

$$\tilde{u}(r) = \frac{1}{r} v(r) = A\frac{\sin kr}{r} + B\frac{\cos kr}{r}$$

If one analyzes the behavior for $r \to 0$, then $A = 0$ and taking a complex combination of v one defines

$$G_k(x, y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}.$$

Note that one actually has a choice to put a minus sign in the exponential function, that is $\frac{1}{4\pi}\frac{e^{-ik|x-y|}}{|x-y|}$ is also a fundamental solution. The next section will justify our choice of sign.

2.3 Sommerfeld Radiation condition

We will be concerned with the exterior Dirichlet problem $-\Delta u - k^2 u = 0$ in Ω^c and $u = g$ in $\partial\Omega$ for a bounded domain Ω (details later). The solution will in general not be unique. However, we can impose additional properties how u should behave at infinity.

Consider the spherical waves $A\frac{e^{ikr}}{r}$ and $B\frac{e^{-ikr}}{r}$. They are solutions of the (radial) Helmholtz equation. Remember that $A\frac{e^{ikr}}{r}e^{-iwt} = A\frac{e^{ik(r-ct)}}{r}$ is then a solution of the wave equation and its phase $r - ct$ is outgoing for $r > 0$. Similiar, the other one will have "left going phase" and hence be incoming.

Let $u_\infty(r) := A\frac{e^{ikr}}{r}$. Then

$$\frac{\partial}{\partial r} u_\infty(r) = A\frac{e^{ikr}}{r}(ik - \frac{1}{r}),$$

and assumung k real,

$$\left| \frac{\partial}{\partial r} u_\infty - iku_\infty \right| = o(\frac{1}{r^2}) \text{ as } r \to \infty.$$

We will demand that a solution of the EDP satisifies the latter property, i.e. behaves like an outgoing wave. It is called the Sommerfeld's radiation condition. Note that $B\frac{e^{-ikr}}{r}$ does not fulfill the condition. Consider $-\Delta u - k^2 u = f$ in $\mathbb{R}^n$. Then $u = G_k * f$ is a solution that behaves likes, for $x = r\omega$

$$A(\omega)\frac{e^{ikr}}{r} + o(r^{-2}) \text{ as } r \to \infty$$

And thus our choice for the sign of the fundamental solution.

We will show that this forces the solution to be unique, and we will be able to talk about "the" radiating solution of the EDP.

2.4 Rellich Lemma and Uniqueness of the EDP

Lemma 2.1. *(Rellich)*
Let $k > 0$ and u a solution of

$$-\Delta u - k^2 u = 0 \textit{ on } \overline{B_{\rho_0}(0)}^c$$

and suppose that

$$\lim_{\rho\to\infty}\int_{|x|=\rho}|u(x)|^2 d\sigma = 0$$

then $u = 0$ *on* $\overline{B_{\rho_0}(0)}^c$.

Proof. (Idea) We have that u is an eigenfunction of Δ outside the closed ball of radius ρ_0. The elliptic regularity theorem tells us that u is thus C^∞. So we can write u as a sum of surface spherical harmonics $\{\mathcal{H}(S^2)_m\}_{m\in\mathbb{N}}$ (remember: these are harmonic homogeneous polynomials (in the three coordinate variables) restricted to the unit surface) which form are dense in $L(S^2)$ and rescaling the unit sphere to ∂B_ρ for $\rho > \rho_0$, $x = \rho\omega$, $|x| = \rho$ and $\omega \in S^2$. So, let the spherical harmonics $\{\psi_{mp}\}$ be an ONB of L^2 and $f_{mp}(k\rho) = \langle u(\rho\,\cdot\,), \psi_{mp}\rangle_{L^2(S^2)}$:

$$u = \sum_{m=0}^{\infty}\sum_{p=1}^{N(n,m)} f_{mp}(k\rho)\psi_{mp}(\omega).$$

Since u satisfies the Helmholtz equation one can show that each $z^{n/2-1}f_{mp}(z)$ is then a solution of a Bessel equation

$$g''(z) + \frac{1}{z}g'(z) + (1 - \frac{(m+\frac{1}{2})^2}{z^2})g(z) = 0.$$

The spherical Hankel functions h_m^1, h_m^2 are a basis of the solution space of this equation. They satisfy

$$h_m^1 = \frac{1}{z^{(n-1)/2}}\left[\exp i\left[z - (2m+2)\pi/4\right] + O(\frac{1}{z})\right]$$

and

$$h_m^2 = \frac{1}{z^{(n-1)/2}}\left[\exp -i\left[z - (2m+2)\pi/4\right] + O(\frac{1}{z})\right]$$

So f_{mp} can be represented as a linear combination of those two, say $f_{mp} = ah_m^1 + bh_m^2$ and hence

$$\rho^2|f_{mp}(k\rho)|^2 = |a\exp 2i\left[k\rho - (2m+2)\pi/4\right] + b|^2 + O(\frac{1}{\rho}).$$

But

$$\int_{|x|=\rho}|u(x)|^2 d\sigma = \sum_{m=0}^{\infty}\sum_{p=1}^{N(n,m)}\rho^2|f_{mp}(k\rho)|^2.$$

Taking the limit $\rho\to\infty$ shows $a = b = 0$, so every f_{mp} vanishes and therefore does u. □

Corollary 2.2. Let $u \in H^1_{loc}(\Omega^c)$ be a solution of the homogeneous exterior Helmholtz equation,

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$

and satisfies in addition to the Sommerfeld radiation condition also

$$\Im\left(k\int_\Gamma(\gamma_N^c\bar{u})(\gamma_D^c u)\,d\sigma\right) \geq 0,$$

then $u = 0$ on Ω^c.

Proof. First note that by the elliptic regularity theorem u is smooth on Ω^c. For what follows it does not matter if we enlarge to some Ω_* such that $\Gamma_* \subset \Omega^c$. So u is smooth in the closure of Ω^c_*. We know lose the asterix. Hence $\gamma_N u = n \cdot \nabla u = \frac{\partial u}{\partial \rho}$ on the ∂B_ρ and so

$$\left|\frac{\partial u}{\partial \rho} - iku\right|^2 = \left|\frac{\partial u}{\partial \rho}\right|^2 + |k|^2|u|^2 + 2\Im\left(k\frac{\partial \bar{u}}{\partial \nu}u\right).$$

Set $\Omega^c_\rho = \Omega^c \cap B_\rho$ and apply Greens first identiy,

$$\int_{\Omega^c_\rho} \nabla\bar{u} \cdot \nabla v - \bar{k}^2\bar{u}v \, dx = \langle(-\Delta - k^2)u, v\rangle_{L^2(\Omega^c_\rho)} + \langle\gamma_N u, \gamma_D v\rangle_{L^2(\partial\Omega^c_\rho)}.$$

Take $v = u$. The first integral on the right hand side vanishes by assumption. Now multiply with k and take the imaginary part to obtain

$$\Im(k)\int_{\Omega^c_\rho} |\nabla u|^2 - |k|^2|u|^2 \, dx = \int_{\partial B_\rho} \Im\left(k\frac{\partial \bar{u}}{\partial \nu}u\right) d\sigma - \int_\Gamma \Im\left(k\frac{\partial \bar{u}}{\partial \nu}u\right) d\sigma.$$

Substitute $\Im\big(k\frac{\partial \bar{u}}{\partial \nu}u\big)$ of the first integral with the equality we already obtained:

$$\Im(k)\int_{\Omega^c_\rho} |\nabla u|^2 + |k|^2|u|^2 \, dx + \frac{1}{2}\int_{\partial B_\rho} \left|\frac{\partial u}{\partial \rho}\right|^2 + |k|^2|u|^2 \, d\sigma =$$
$$+\int_{\partial B_\rho} \left|\frac{\partial u}{\partial \rho} - iku\right|^2 d\sigma - \int_\Gamma \Im\left(k\frac{\partial \bar{u}}{\partial \nu}u\right) d\sigma.$$

So if $\Im(k) > 0$ and using the assumption, the LHS is positive whereas by the Sommerfeld radiation the right hand side converges to something ≤ 0 as $\rho \to \infty$. Therefore $\int_{\Omega^c_\rho} |u|^2 dx \to 0$ and u must vanish on Ω^c.
If we assume k to be real, we can merely conclude ($|k|^2 > 0!$) that $\int_{\partial B_\rho} |u|^2 dx \to 0$. But now Rellich Lemma applies.

□

Theorem 2.3. *(Uniqueness)*
There is at most one radiating solution $u \in H^1_{loc}(\Omega^c)$ *of*

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$
$$\gamma^c_D u = g \text{ on } \Gamma,$$

where $g \in H^{\frac{1}{2}}(\Gamma)$.

Proof. If v, w are two solutions, then $u = v - w$ satisfies the assumption of the corrolary since $\gamma_D u = 0$. □

3 BIE of Helmholtz Equation

3.1 Integral operators and Boundary Integral Equations

We introduce now the integral representations of the solutions of the Helmholtz equation. In particular we will give a proof of the existence of a solution of the EDP.

Recall the fundamental solution of the Helmholtz Equation

$$-\Delta u(x) - k^2 u(x) = 0 \text{ for } x \in \Omega \subset \mathbb{R}^3,$$

namely

$$G_k(x,y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}.$$

The standard boundary integral operators for $x \in \Gamma$ are then

$$\begin{aligned}
\text{Weakly Singular Boundary Integral Operator } (V_k w)(x) &= \int_\Gamma G_k(x,y) w(y) ds_y \\
\text{Double Layer Potential } (K_k v)(x) &= \int_\Gamma \frac{\partial}{\partial n_y} G_k(x,y) v(y) ds_y \\
\text{Adjoint Double Layer Potential } (K_k' v)(x) &= \int_\Gamma \frac{\partial}{\partial n_x} G_k(x,y) v(y) ds_y \\
\text{Hypersingular Boundary Integral Operator } (W_k v)(x) &= -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} G_k(x,y) v(y) ds_y
\end{aligned}$$

Theorem 3.1. *For a bounded Lipschitz domain, the boundary integral operators*

$$\begin{aligned}
V_k &: H^{-\frac{1}{2}+s}(\Gamma) \to H^{\frac{1}{2}+s}(\Gamma) \\
K_k &: H^{\frac{1}{2}+s}(\Gamma) \to H^{\frac{1}{2}+s}(\Gamma) \\
K_k' &: H^{-\frac{1}{2}+s}(\Gamma) \to H^{-\frac{1}{2}+s}(\Gamma) \\
W_k &: H^{\frac{1}{2}+s}(\Gamma) \to H^{-\frac{1}{2}+s}(\Gamma)
\end{aligned}$$

are bounded for all $s \in [-\frac{1}{2}, \frac{1}{2}]$

However the ellipticity property of V_k no longer holds. We now rather prove

Lemma 3.2. $V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ *is coercive, i.e. there exists a compact operator* $C : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ *such that the Gardings inequality*

$$\langle V_k w, w\rangle_\Gamma + \langle Cw, w\rangle_\Gamma \geq \text{const } ||w||^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall w \in H^{-\frac{1}{2}}(\Gamma)$$

holds.

Proof. The idea is the following: Let $C = \delta V = V_0 - V_k$. Then $V_0 = V_k + \delta V$. Consider $u = \Psi_{SL} w - \Psi^0_{SL} w$. One calculates $-\Delta[-\Delta - k^2]u = 0$ and can therefore argue that $\Psi_{SL} - \Psi^0_{SL} : H^{-\frac{1}{2}}(\Gamma) \to H^3(\Omega)$ and thus $\delta V = \gamma_D(V - V_0) : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{5}{2}}(\Gamma)$. Since the embedding of $H^{\frac{5}{2}}$ in $H^{\frac{1}{2}}$ is compact, $\delta V : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}$ is also compact. The claim then follows from the $H^{-\frac{1}{2}}$-ellipticity of V_0. □

Note that also $W_k - W_0$, $K_k - K_0$ and $K_k' - K_0'$ are compact.

Theorem 3.3. *Let* $\phi \in H^{-\frac{1}{2}}(\Gamma)$. *Then* $u = \Psi_{SL}\phi$ *satisfies the homogeneous partial differential equation* $(-\Delta - k^2)u = 0$ *in* $\mathbb{R}^n \setminus \Gamma$. *It also satifies the Sommerfeld radiation condition. Furthermore* $[\gamma_D u] = 0$ *Analogously, for* $\phi \in H^{\frac{1}{2}}(\Gamma)$. *Then if* $u = \Psi_{DL}\phi$ *we have* $(-\Delta - k^2)u = 0$ *in* $\mathbb{R}^n \setminus \Gamma$ *and also satifies the Sommerfeld radiation condition. Furthermore* $[\gamma_N u] = 0$

Remark 3.4. One has the equalities $W_k = K_k + \frac{1}{2}\,\mathrm{Id}$ and $V_k = K'_k + \frac{1}{2}\,\mathrm{Id}$.

Suppose that $u \in H^1_{loc}(\Omega^c)$ with $(-\Delta - k^2)u = 0$ in Ω^c satisfies $u = -\Psi_{SL}[\gamma_D u] + \Psi_{DL}[\gamma_D u]$ in Ω^c. Taking the Dirichlet trace and Neumann trace repectively and applying above identies one has

$$\begin{pmatrix} \gamma_D^c u \\ \gamma_N^c u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K_k & -V_k \\ -W_k & \frac{1}{2}I - K'_k \end{pmatrix} \begin{pmatrix} \gamma_D^c u \\ \gamma_N^c u \end{pmatrix}$$

Note the change of signs with respect to the interior Calderón Operator.

We now list the corresponding boundary integral equations to the exterior Dirichlet problem.

Direct Method - integral equality of the first kind: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\langle V_k\psi, \varphi\rangle_{L^2(\Gamma)} = -\frac{1}{2}\langle g, \varphi\rangle_{L^2(\Gamma)} + \langle K_k g, \varphi\rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma)$$

Direct Method - integral equality of the second kind: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\frac{1}{2}\langle \psi, \varphi\rangle_{L^2(\Gamma)} + \langle K'_k\psi, \varphi\rangle_{L^2(\Gamma)} = -\langle W_k g, \varphi\rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma)$$

Indirect Method - using the single layer potential: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\langle V_k\psi, \varphi\rangle_{L^2(\Gamma)} = \langle g, \varphi\rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma)$$

Indirect Method - using the double layer potential: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\frac{1}{2}\langle \psi, \varphi\rangle_{L^2(\Gamma)} + \langle K_k\psi, \varphi\rangle_{L^2(\Gamma)} = \langle g, \varphi\rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma)$$

3.2 Representation formula

We have seen Green's representation formula that states if Ω is bounded in $u \in H^1(\Omega)$ is a solution to the homogeneous interior Dirichlet problem for the Laplace. It also holds for the Helmholtz operator, i.e. if $(-\Delta - k^2)u = 0$, $\gamma_D u = g$, $g \in H^{\frac{1}{2}}(\Gamma)$

$$u = \Psi_{SL}(\gamma_N u) - \Psi_{DL}(\gamma_D u) \text{ in } \Omega.$$

A more general version of it goes as follows (with the closed brackets denoting the jumps of the traces):

Theorem 3.5. *$u \in L^2(\mathbb{R}^n)$ with compact support, $u|_\Omega \in H^1(\Omega)$, $u|_{\Omega^c} \in H^1(\Omega^c)$, then*

$$u = \Psi_{DL}([\gamma_D u]) - \Psi_{SL}([\gamma_N u]) \text{ on } \mathbb{R}^n$$

Obviously, for the EDP we don't want to restrict solutions to have compact support but satisfy only the Sommerfeld radiating condition, writing $x = \rho\omega, \rho = |x|, \omega \in \mathbb{S}^2$:

$$\lim_{\rho\to\infty} \rho \left(\frac{\partial u}{\partial r} - iku \right) = 0$$

uniformely in ω.

Remarkably, the representation formula still holds:

Theorem 3.6. *Let $g \in H^{\frac{1}{2}}(\Gamma)$. And suppose $u \in H^1_{loc}(\Omega^c)$ is a radiating solution of*

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$
$$\gamma_D^c u = g \text{ on } \Gamma,$$

then u has the integral representation

$$u = \Psi_{DL} g - \Psi_{SL}(\gamma_N^c u).$$

Proof. We sketch the proof. Once more, put $\Omega^c_\rho = \Omega^c \cap B_\rho$. The boundedness of this domain implies $u \in H^1(\Omega^c_\rho)$ (remember the definition of $H^l_{loc}(W) = \{t \in \mathcal{D}'(W) : t \in H^1(B) \text{ for all bounded } B \subset W\}$) and that we can apply Greens representation theorem

$$u(x) = +\Psi_{SL}(\gamma_N^c u)(x) - \Psi_{DL}(\gamma_D^c u)(x) \\ - \int_{\partial B_\rho} G(x,y)\gamma_N u(y)\, d\sigma + \int_{\partial B_\rho} \gamma_N G(x,y)\gamma_D u(y)\, d\sigma \text{ in } \Omega^c_\rho \quad (\star),$$

the second line corresponding to the "outer" boundary. Note the different signs: On Γ, we use the outer normal unit vector of Ω which therefore looks inward to $\Omega^c_\rho = \Omega^c \cap B_\rho$ and so Ψ_{SL} carries a $+$ in front of it and Ψ_{DL} a $-$. On ∂B_ρ we do consider the outward UNV.

Remember the uniqueness result? We showed that the Sommerfeld radiation condition implied that

$$\Im(k) \int_{\Omega^c_\rho} |\nabla u|^2 + |k|^2 |u|^2\, dx + \frac{1}{2} \int_{\partial B_\rho} \left|\frac{\partial u}{\partial \rho}\right|^2 + |k|^2|u|^2\, d\sigma \\ \to - \int_\Gamma \Im\left(k \frac{\partial \bar{u}}{\partial \nu} u\right) d\sigma \text{ as } \rho \to \infty.$$

In particular, $\int_{\partial B_\rho} |u|^2\, d\sigma$ has to be bounded. Note that $\int_{\partial B_\rho} |G(x,y)|^2\, d\sigma$ is also bounded and the fundamental solution is also radiating. We then write

$$\int_{\partial B_\rho} G(x,y)\gamma_N u(y)\, d\sigma + \int_{\partial B_\rho} \gamma_N G(x,y)\gamma_D u(y)\, d\sigma = \\ \int_{\partial B_\rho} G(x,y)\left(\gamma_N u(y) - iku(y)\right)\, d\sigma + \int_{\partial B_\rho} \left(\gamma_N G(x,y) - ikG(x,y)\right)\gamma_D u(y)\, d\sigma$$

We apply on both integrals Cauchy-Schwarz, the former then vanishes by the radiating condition on u as $\rho \to \infty$, the latter because of the radiating property of G. Finally, let $\rho \to \infty$ in $(\star)$ and the claim follows. □

3.3 Existence

We finally will prove the existence of a radiating solution. Of utmost importance is the following correspondence between the volume problem and the boundary integral equation:

Theorem 3.7. *Let $g \in H^{\frac{1}{2}}(\Gamma)$. And suppose $u \in H^1_{loc}(\Omega^c)$ is a radiating solution of*

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$
$$\gamma_D^c u = g \text{ on } \Gamma,$$

then $\phi = \gamma_N^c \in H^{-\frac{1}{2}}(\Gamma)$ *is a solution of the boundary integral equation*

$$V_k\phi = (-\operatorname{Id}/2 + K)g \text{ on } \Gamma, \tag{1}$$

and u *has the integral representation*

$$u = \Psi_{DL}g - \Psi_{SL}\phi. \tag{2}$$

Conversely, if $\phi \in H^{-\frac{1}{2}}(\Gamma)$ *is a solution (1), then formula (2) defines a solution* $u \in H^1_{loc}(\Omega^c)$ *of the exterior Dirichlet problem.*

Proof. First suppose u to be a solution of the EDP. We already proved the representation formula $u = \Psi_{DL}(\gamma_D^c) - \Psi_{SL}(\gamma_N^c u)$. Take the Dirichlet trace to get

$$\gamma_D^c u = -V_k\gamma_N^c u + (K + \frac{1}{2}\operatorname{Id})\gamma_D^c u.$$

But with $\phi = \gamma_N^c u$, $g = \gamma_D^c u$ this is just (1).

Now, let $u := \Psi_{DL}g - \Psi_{SL}\phi$ where ϕ is a solution of the BIE (1). We know that $(-\Delta - k^2)\Psi_{DL} \equiv 0$ and $(-\Delta - k^2)\Psi_{SL} \equiv 0$ and both $\Psi_{DL}v$ and $\Psi_{SL}w$ satisfy the Sommerfeld radiation condition, $v \in H^{\frac{1}{2}}(\Gamma)$, $w \in H^{-\frac{1}{2}}(\Gamma)$. Hence, we only need to check that $\gamma_D u = g$. But again, taking the trace of u gives

$$\gamma_D^c u = (K + \frac{1}{2}\operatorname{Id})g - V_k\phi = (K + \frac{1}{2}\operatorname{Id})g - (-\frac{1}{2}\operatorname{Id} + K)g = g$$

and u solves the EDP. To proof that $u \in H^1_{loc}(\Omega^c)$ one needs the fact that for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ it holds $\varphi\Psi_{SL} : H^{-\frac{1}{2}}(\Gamma) \to H^1(\mathbb{R}^n)$ and $\varphi\Psi_{DL} : H^{\frac{1}{2}}(\Gamma) \to H^1(\Omega^c)$ and that elements of $H^1_{loc}(W)$ can also be characterized by

$$v \in H^1_{loc}(W) \text{ iff } \varphi v \in H^1(\mathbb{R}^n) \forall \varphi \in \mathcal{D}(W)$$

□

Theorem 3.8. V_k *is injective on* $H^{-\frac{1}{2}}(\Gamma)$ *iff* k^2 *is not an eigenvalue of* $-\Delta$ *of the interior Dirichlet problem, i.e.*

$$-\Delta u = k^2 u \text{ on } \Omega, \quad \gamma_D u = 0 \Rightarrow u = 0 \text{ on } \Omega$$

The kernel of V_k *is given by*

$$\ker(V_k) = \operatorname{span}\left[\gamma_N v : -\Delta v = k^2 v \text{ on } \Omega \text{ and } \gamma_D v = 0 \text{ on } \Gamma\right]$$

Proof. Suppose v satisfies $-\Delta v = k^2 v$ and $\gamma_D v = 0$. So, by Green's representation formula for v, we have $v = \Psi_{SL}\gamma_N v$ in Ω. Apply the interior Dirichlet trace to get

$$0 = \gamma_D v = \gamma_D\Psi_{SL}\gamma_N v = V_k\gamma_N v.$$

Therefore $\gamma_N v \in \ker(V_k)$. Conversely, let $w \in \ker(V_k)$. Define $v = \Psi_{SL}w$. The single layer potential satisfies $L_k\Psi_{SL} = 0$, so $-\Delta v = k^2 v$. Also $0 = V_k w = \gamma_D\Psi_{SL}w = \gamma_D v$. Hence if k^2 is not to be an eigenvalue of $-\Delta$ of the above IDP, v must be indentically zero proving that the kernal is trivial. □

Theorem 3.9. *(Fredholm Alternative)*
Let $A \in \mathcal{B}(X, Y)$ coercive. If $Au = 0$ only allows the trivial solution $u = 0$, then $Au = f$ is uniquely solvable for all $f \in Y$. Else $Au = f$ is solvable iff $\langle v, f\rangle = 0$ for all $v \in Y^* : A^*v = 0$.

Theorem 3.10. *(Existence and Uniqueness)*
Let Ω be a bounded Lipschitz domain with boundary Γ. Then for every $g \in H^{\frac{1}{2}}(\Gamma)$ the exterior Dirchlet problem

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$
$$\gamma_D^c u = g \text{ on } \Gamma$$

has a unique solution $u \in H^1_{loc}(\Omega^c)$ that satisfies the Sommerfeld radiation condition.

Proof. The uniqueness readily follows from the corollary of the Rellich lemma. To prove existence, we will solve the boundary integral equation $V_k\phi = (-\operatorname{Id}/2 + K_k)g$. By the above theorem u defined as $u = \psi_{DL}g - \psi_{SL}\phi$ will then be a radiating solution. We use the Fredholm alternative: We already showed coercivity of V_k. If k^2 is not an eigenvalue we also have injectivity and therefore V_k is invertible and the BIE is solvable (even uniquely!). If k^2 is an eigenvalue, we have to check the solvability condition $\langle w, f\rangle = 0$ for all $w \in \ker(V_k^*)$ and f the right hand side of the BIE. But we can apply the theorem about injectivity of V_k to see that

$$\ker(V^*) = \operatorname{span}\left[\gamma_N v : v \in H^1(\Omega),\ -\Delta v = \overline{k}^2 v \text{ on } \Omega \text{ and } \gamma_D v = 0 \text{ on } \Gamma\right]$$

Apply the second of Green's identities to get

$$\begin{aligned}\langle \gamma_N v, (-\frac{1}{2}I + K_k)g\rangle_{L^2(\Gamma)} &= \langle \gamma_N v, \gamma_D(\Psi_{DL}g)\rangle_{L^2(\Gamma)} \\ &= \langle \gamma_D v, \gamma_N(\Psi_{DL}g)\rangle_{L^2(\Gamma)} \\ &- \langle(-\Delta - \bar{k}^2)v, \Psi_{DL}g\rangle_{L^2(\Omega)} + \langle v, (-\Delta - k^2)\Psi_{DL}g\rangle_{L^2(\Omega)}\end{aligned}$$

Each of the three terms on the right vanishes since $\gamma_D v = 0$ on Γ, $(-\Delta - \bar{k}^2)v = 0$ and $(-\Delta - k^2)\Psi_{DL}g = 0$ on Ω.

□

Note that the BIE in the case of resonance is not uniquely solvable even though the exterior Dirchlet problem is (assuming the radiating condition)!

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