The Helmholtz Equation

Seminar BEM on Wave Scattering

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Outline

- Steklov-Poincare Operator
- Helmholtz Equation: From the Wave equation to Radiation condition
- Uniqueness of the Exterior Dirichlet Problem
- Boundary Integral Operators
- Representation formula
- Existence of the Exterior Dirichlet Problem

Poincare - Steklov Operator

Last time we saw the Calderon projection, enabling us to write the following system of BIE for the Cauchy data $\gamma_D u, \gamma_N u$

$$\begin{pmatrix} \gamma_D u \\ \gamma_N u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I - K_0 & V_0 \\ W_0 & \frac{1}{2} I + K_0' \end{pmatrix} \begin{pmatrix} \gamma_D u \\ \gamma_N u \end{pmatrix}$$

(here for for homogeneous case $\Delta u = 0$) where

$$\begin{aligned} V_0 &: H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \\ K_0 &: H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \\ K_0' &: H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \\ W_0 &: H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \end{aligned}$$

Steklov - Poincare Operator

- We proved (in 3d) $V_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is elliptic $\Rightarrow$ invertible!

- Rewrite first equation:

$$\gamma_N u = V_0^{-1}(\frac{1}{2}I + K_0)\gamma_D u$$

- Define $S_0 := V_0^{-1}(\frac{1}{2}I + K_0) : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$
- ...called Steklev-Poincare-Operator

$\rightarrow$ S_0 is a Dirichlet-to-Neumann map

Steklov - Poincare Operator

- Second equation of the Calderon projection admits another representation:

$$S_0 = W_0 + (\frac{1}{2}I + K_0')V_0^{-1}(\frac{1}{2}I + K_0)$$

- S_0 admits the same ellipticity estimates as W_0:

$$\langle Sv, v \rangle_\Gamma \geq \langle W_0 v, v \rangle_\Gamma$$

- $\Rightarrow H_*^{\frac{1}{2}}(\Gamma)$-elliptic.

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Wave equation reduces to Helmholtz Equation

- The wave equation is

$$\frac{\partial^2}{\partial t^2}\Psi - c^2\Delta\Psi = 0$$

- assume the solution to be time harmonic:

$$\Psi(t,x) = e^{-iwt}u(x)$$

- then u will satisfy

$$-w^2u - c^2\Delta u = 0 \text{ or } -\Delta u - k^2u = 0 \text{ with } k = \frac{w}{c}$$

- ... called Helmholtz equation.

Wave equation reduces to Helmholtz Equation

- Fundamental solution of $-\Delta u - k^2 u = 0$ in $\mathbb{R}^3$

$$G_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} \quad x \neq y$$

- also $\frac{1}{4\pi}\frac{e^{-ik|x-y|}}{|x-y|}$ is a fundamental solution - justify our choice later
- Just as for the Laplace, in n=2, things are a little bit different, Y_0 the second kind Bessel function of order zero,

$$G_k(x,y) = \frac{1}{2\pi} Y_0(k|x-y|) \quad x \neq y$$

- whose singularity at $x \to y$ behaves like the $\log s$ as $s \to 0$
- we study the Exterior Dirichlet Problem (EDP) for a bounded domain $\Omega \subset \mathbb{R}^3$

$$-\Delta u - k^2 u = 0 \text{ in } \Omega^c \text{ and } u = g \text{ in } \partial\Omega = \Gamma$$

Radiation condition

- Problem: in general won't be unique
- ... for consider $A\frac{e^{ikr}}{r}$ and $B\frac{e^{-ikr}}{r}$ spherical waves will both satisfy the Helmholtz equation
- want our solution of the EDP also behaves like an outgoing wave!
- so let's look for a condition only $u_\infty = A\frac{e^{ikr}}{r}$ satisfies.

$$\left| \frac{\partial}{\partial r} u_\infty - iku_\infty \right| = o(\frac{1}{r^2}) \text{ as } r \to \infty$$

Sommerfeld Radiation Condition!

Radiation condition

- consider $-\Delta u - k^2 u = f$ in $\mathbb{R}^n$.
- $u = G_k * f$ is a solution
- .. that satifies, putting $x = r\omega$

$$A(\omega)\frac{e^{ikr}}{r} + o(r^{-2}) \text{ as } r \to \infty$$

- right choice of sign!

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Uniqueness

Uniqueness of the EDP!

Theorem

There is at most one radiating solution $u \in H^1_{loc}(\Omega^c)$ *of*

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$
$$\gamma_D^c u = g \text{ on } \Gamma,$$

where $g \in H^{\frac{1}{2}}(\Gamma)$.

Uniqueness

- need the following Lemma established by Rellich:

Lemma

Let $k > 0$ *and* u *a solution of*

$$-\Delta u - k^2 u = 0 \text{ on } \overline{B_{\rho_0}(0)}^c$$

and suppose that

$$\lim_{\rho \to \infty} \int_{|x|=\rho} |u(x)|^2 d\sigma = 0$$

then $u = 0$ *on* $\overline{B_{\rho_0}(0)}^c$.

Uniqueness

- and, with a little work, it follows that

Corollary

Let $u \in H^1_{loc}(\Omega^c)$ be a solution of the homogeneous exterior Helmholtz equation,

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$

and satisfies in addition to the Sommerfeld radiation condition also

$$\Im\left(k \int_\Gamma (\gamma_N^c \bar{u})(\gamma_D^c u)\, d\sigma\right) \geq 0,$$

then $u = 0$ on Ω^c.

Uniqueness

Proof:

- Let's bound our domain: $\Omega_\rho^c = \Omega^c \cap B_\rho$

$$\overset{\text{Green's 1st}}{\Rightarrow} \int_{\Omega_\rho^c} \nabla \bar{u} \cdot \nabla u - \bar{k}^2 \bar{u} u \, dx = \langle (-\Delta - k^2) u, u \rangle_{L^2(\Omega_\rho^c)} + \langle \gamma_N u, \gamma_D u \rangle_{L^2(\partial \Omega_\rho^c)}$$

- first term vanishes by assumption. Multiply with k and take the imaginary part

$$\Im(k) \int_{\Omega_\rho^c} |\nabla u|^2 - |k|^2 |u|^2 \, dx = \int_{\partial B_\rho} \Im\left(k \frac{\partial \bar{u}}{\partial \nu} u \right) d\sigma - \int_\Gamma \Im\left(k \frac{\partial \bar{u}}{\partial \nu} u \right) d\sigma$$

Uniqueness

- last integral can be rewritten since

$$\left|\frac{\partial u}{\partial \rho} - iku\right|^2 = \left|\frac{\partial u}{\partial \rho}\right|^2 + |k|^2|u|^2 + 2\Im\left(k\frac{\partial \bar{u}}{\partial \nu}u\right)$$

$$\Rightarrow \Im(k)\int_{\Omega_\rho^c} |\nabla u|^2 + |k|^2|u|^2\, dx + \frac{1}{2}\int_{\partial B_\rho} \left|\frac{\partial u}{\partial \rho}\right|^2 + |k|^2|u|^2\, d\sigma = \\ + \int_{\partial B_\rho} \left|\frac{\partial u}{\partial \rho} - iku\right|^2\, d\sigma - \int_\Gamma \Im\left(k\frac{\partial \bar{u}}{\partial \nu}u\right)\, d\sigma.$$

- $\Im(k) > 0 \Rightarrow \int_{\Omega_\rho^c} |u|^2 dx \to 0$
- $\Im(k) = 0 \Rightarrow$ Rellich applies

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Integral operators and Boundary Integral Equations

- As for the Laplace, we define the boundary integral operators as

$$(V_k w)(x) = \int_\Gamma G_k(x,y) w(y) ds_y$$
$$(K_k v)(x) = \int_\Gamma \frac{\partial}{\partial n_y} G_k(x,y) v(y) ds_y$$
$$(K_k' v)(x) = \int_\Gamma \frac{\partial}{\partial n_x} G_k(x,y) v(y) ds_y$$
$$(W_k v)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} G_k(x,y) v(y) ds_y$$

Integral operators and Boundary Integral Equations

- with the same properties as for the Laplace

Theorem

For a bounded Lipschitz domain, the boundary integral operators

$$V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$$
$$K_k : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$$
$$K_k' : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$$
$$W_k : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$$

are continous.

Remark

One has the equalities $W_k = K_k + \frac{1}{2}\,\mathrm{Id}$ *and* $V_k = K_k' + \frac{1}{2}\,\mathrm{Id}$.

Integral operators and Boundary Integral Equations

- well... not exactly the same properties: V_k no longer elliptic!
- but we still have

Lemma

$V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is coercive, i.e. there exists a compact operator $C : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ such that the Gardings inequality

$$\langle V_k w, w \rangle_{L^2(\Gamma)} + \langle Cw, w \rangle_{L^2(\Gamma)} \geq \text{const } ||w||^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \forall w \in H^{-\frac{1}{2}}(\Gamma)$$

holds.

- the single and double layer potential give us radiating solution:

Theorem

If $\phi \in H^{-\frac{1}{2}}(\Gamma)$ then for $u = \Psi_{SL}\phi$ satisfies the Sommerfeld radiation condition and

$$(-\Delta - k^2)u = 0 \text{ in } \mathbb{R}^n \setminus \Gamma$$

- Analogously for $u = \Psi_{DL}\phi$ if $\phi \in H^{\frac{1}{2}}(\Gamma)$.

Integral operators and Boundary Integral Equations

- suppose $u \in H^1_{loc}(\Omega^c)$ with $(-\Delta - k^2)u = 0$ in $\Omega \cup \Omega^c$ satisfies

$$u = -\Psi_{SL}[\gamma_N u] + \Psi_{DL}[\gamma_D u] \text{ in } \Omega \cup \Omega^c$$

$\rightarrow$ Integral representation formula

- take traces to get the exterior Calderon projection

$$\begin{pmatrix} \gamma_D^c u \\ \gamma_N^c u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K_k & -V_k \\ -W_k & \frac{1}{2}I - K'_k \end{pmatrix} \begin{pmatrix} \gamma_D^c u \\ \gamma_N^c u \end{pmatrix}$$

- different signs w.r.t to the interior Calderon

Integral operators and Boundary Integral Equations

- boundary integral equations to the exterior Dirichlet problem:
- Direct Method - integral equality of the first kind: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\langle V_k \psi, \varphi \rangle_{L^2(\Gamma)} = -\frac{1}{2} \langle g, \varphi \rangle_{L^2(\Gamma)} + \langle K_k g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma)$$

- Direct Method - integral equality of the second kind: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\frac{1}{2} \langle \psi, \varphi \rangle_{L^2(\Gamma)} + \langle K_k' \psi, \varphi \rangle_{L^2(\Gamma)} = -\langle W_k g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma)$$

Integral operators and Boundary Integral Equations

- Indirect Method - using the single layer potential: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\langle V_k \psi, \varphi \rangle_{L^2(\Gamma)} = \langle g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma)$$

- Indirect Method - using the double layer potential: Let $g \in H^{\frac{1}{2}}(\Gamma)$. Find $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\frac{1}{2}\langle \psi, \varphi \rangle_{L^2(\Gamma)} + \langle K_k \psi, \varphi \rangle_{L^2(\Gamma)} = \langle g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma)$$

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Representation Formula

- for the exterior Calderon operator we assumed Greens representation formula.. does it hold?
- yes - thanks to the radiating property!

Theorem

Let $g \in H^{\frac{1}{2}}(\Gamma)$. And suppose $u \in H^1_{loc}(\Omega^c)$ is a radiating solution of

$$-\Delta u - k^2 u = 0 \ on \ \Omega^c$$
$$\gamma_D^c u = g \ on \ \Gamma,$$

then u has the integral representation

$$u = \Psi_{DL} g - \Psi_{SL}(\gamma_N^c u).$$

Representation Formula

Proof idea:

- $\Omega_\rho^c = \Omega^c \cap B_\rho$

$$\overset{Green}{\Rightarrow} u(x) = +\Psi_{SL}(\gamma_N^c u)(x) - \Psi_{DL}(\gamma_D^c u)(x) \\ - \int_{\partial B_\rho} G(x,y)\gamma_N u(y)\, d\sigma + \int_{\partial B_\rho} \gamma_N G(x,y)\gamma_D u(y)\, d\sigma$$

- Sommerfeld radiation condition $\Rightarrow$ two last terms vanish as $\rho \to 0$

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Existence

- we're ready to prove the existence!
- ..by showing that we can solve the BIE.
- so we need

Theorem

Let $g \in H^{\frac{1}{2}}(\Gamma)$. And suppose $u \in H^1_{loc}(\Omega^c)$ is a radiating solution of

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c \quad \gamma_D^c u = g \text{ on } \Gamma,$$

then $\phi = \gamma_N^c \in H^{-\frac{1}{2}}(\Gamma)$ is a solution of the boundary integral equation

$$V_k \phi = (-\operatorname{Id}/2 + K) g \text{ on } \Gamma, \tag{1}$$

and u has the integral representation

$$u = \Psi_{DL} g - \Psi_{SL} \phi. \tag{2}$$

Conversely, if $\phi \in H^{-\frac{1}{2}}(\Gamma)$ is a solution (1), then formula (2) defines a solution $u \in H^1_{loc}(\Omega^c)$ of the exterior Dirichlet problem.

Existence

- the wave number k has influence on the BIE, namely

Theorem

V_k is injective on $H^{-\frac{1}{2}}(\Gamma)$ iff k^2 is not an eigenvalue of $-\Delta$ of the interior Dirichlet problem, i.e.

$$-\Delta u = k^2 u \text{ on } \Omega, \quad \gamma_D u = 0 \Rightarrow u = 0 \text{ on } \Omega$$

The kernel of V_k is given by

$$\ker(V_k) = \operatorname{span}\left[\gamma_N v : -\Delta v = k^2 v \text{ on } \Omega \text{ and } \gamma_D v = 0 \text{ on } \Gamma\right]$$

Existence

- to finalize, we also need the Fredholm Alternative

Theorem

Let $A \in \mathcal{B}(X, Y)$ coercive. If $Au = 0$ only allows the trivial solution $u = 0$, then $Au = f$ is uniquely solvable for all $f \in Y$. Else $Au = f$ is solvable iff $\langle v, f \rangle = 0$ for all $v \in Y^* : A^* v = 0$.

Existence

- so at last

Theorem

Let Ω be a bounded Lipschitz domain with boundary Γ. Then for every $g \in H^{\frac{1}{2}}(\Gamma)$ the exterior Dirchlet problem

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$
$$\gamma_D^c u = g \text{ on } \Gamma$$

has a unique solution $u \in H^1_{loc}(\Omega^c)$ that satisfies the Sommerfeld radiation condition.

Existence

- need to solve $V_k\phi = (-\operatorname{Id}/2 + K_k)g$ on Γ
- V_k coercive, so apply Fredholm!
- k^2 not an eigenvalue $\overset{\text{Fredholm}}{\Rightarrow}$ V_k invertible
- Else: need to check solvability condition $\langle w, (-\operatorname{Id}/2 + K_k)g\rangle_{L^2(\Gamma)} = 0$ for all $w \in \ker(V_k^*)$
- What's $\ker(V_k^*)$? Can modifiy thm about $\ker(V_k)$!

$$\ker(V^*) = \operatorname{span}\left[\gamma_N v : -\Delta v = \overline{k}^2 v \text{ on } \Omega, \gamma_D v = 0 \text{ on } \Gamma\right]$$

Existence

- so take such a $\gamma_N v$ and apply Green's Identity

$$
\begin{aligned}
\langle \gamma_N v, (-\frac{1}{2} I + K_k) g \rangle_{L^2(\Gamma)} &= \langle \gamma_N v, \gamma_D (\Psi_{DL} g) \rangle_{L^2(\Gamma)} \\
&= \langle \gamma_D v, \gamma_N (\Psi_{DL} g) \rangle_{L^2(\Gamma)} \\
&- \langle (-\Delta - \bar{k}^2) v, \Psi_{DL} g \rangle_{L^2(\Omega)} \\
&+ \langle v, (-\Delta - k^2) \Psi_{DL} g \rangle_{L^2(\Omega)} \\
&= 0
\end{aligned}
$$

- ...since $\gamma_D v = 0$ on Γ, $(-\Delta - \bar{k}^2) v = 0$ and $(-\Delta - k^2)\Psi_{DL} g = 0$ on Ω.