The Helmholtz Equation
Seminar BEM on Wave Scattering

Rene Rühr

ETH Zürich

October 28, 2010
Outline

- Steklov-Poincare Operator
- Helmholtz Equation: From the Wave equation to Radiation condition
- Uniqueness of the Exterior Dirichlet Problem
- Boundary Integral Operators
- Representation formula
- Existence of the Exterior Dirichlet Problem
Last time we saw the Calderon projection, enabling us to write the following system of BIE for the Cauchy data $\gamma_D u, \gamma_N u$

$$
\begin{pmatrix}
\gamma_D u \\
\gamma_N u
\end{pmatrix} = 
\begin{pmatrix}
\frac{1}{2} I - K_0 & V_0 \\
W_0 & \frac{1}{2} I + K'_0
\end{pmatrix}
\begin{pmatrix}
\gamma_D u \\
\gamma_N u
\end{pmatrix}
$$

(here for for homogeneous case $\Delta u = 0$) where

- $V_0 : H^{-\frac{1}{2}}(\Gamma) \to H^\frac{1}{2}(\Gamma)$
- $K_0 : H^\frac{1}{2}(\Gamma) \to H^\frac{1}{2}(\Gamma)$
- $K'_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$
- $W_0 : H^\frac{1}{2}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$
Steklov - Poincare Operator

- We proved (in 3d) $V_0 : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ is elliptic $\Rightarrow$ invertible!

- Rewrite first equation:

$$\gamma_N u = V_0^{-1}(\frac{1}{2}I + K_0)\gamma_D u$$

- Define $S_0 := V_0^{-1}(\frac{1}{2}I + K_0) : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$

- ...called Steklev-Poincare-Operator

$\rightarrow S_0$ is a Dirichlet-to-Neumann map
Second equation of the Calderon projection admits another representation:

$$ S_0 = W_0 + \left( \frac{1}{2} I + K'_0 \right) V_0^{-1} \left( \frac{1}{2} I + K_0 \right) $$

$S_0$ admits the same ellipticity estimates as $W_0$:

$$ \langle Sv, v \rangle_\Gamma \geq \langle W_0 v, v \rangle_\Gamma $$

$$ \Rightarrow H^{\frac{1}{2}}_*(\Gamma)\text{-elliptic.} $$
Outline

- Steklov-Poincare Operator
- Helmholtz Equation: From the Wave equation to Radiation condition
- Uniqueness of the Exterior Dirichlet Problem
- Boundary Integral Operators
- Representation formula
- Existence of the Exterior Dirichlet Problem
Wave equation reduces to Helmholtz Equation

The wave equation is

$$\frac{\partial^2}{\partial t^2} \psi - c^2 \Delta \psi = 0$$

assume the solution to be time harmonic:

$$\psi(t, x) = e^{-i\omega t} u(x)$$

then $u$ will satisfy

$$-\omega^2 u - c^2 \Delta u = 0 \text{ or } -\Delta u - k^2 u = 0 \text{ with } k = \frac{\omega}{c}$$

... called Helmholtz equation.
Wave equation reduces to Helmholtz Equation

- Fundamental solution of $-\Delta u - k^2 u = 0$ in $\mathbb{R}^3$

$$G_k(x, y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} \quad x \neq y$$

- also $\frac{1}{4\pi} \frac{e^{-ik|x-y|}}{|x-y|}$ is a fundamental solution - justify our choice later

- Just as for the Laplace, in $n=2$, things are a little bit different, $Y_0$ the second kind Bessel function of order zero,

$$G_k(x, y) = \frac{1}{2\pi} Y_0(k|x-y|) \quad x \neq y$$

- whose singularity at $x \rightarrow y$ behaves like the log $s$ as $s \rightarrow 0$

- we study the Exterior Dirichlet Problem (EDP) for a bounded domain $\Omega \subset \mathbb{R}^3$

$$-\Delta u - k^2 u = 0 \text{ in } \Omega^c \text{ and } u = g \text{ in } \partial \Omega = \Gamma$$
Radiation condition

- Problem: in general won’t be unique
- ... for consider \( A e^{ikr} \) and \( B e^{-ikr} \) spherical waves will both satisfy the Helmholtz equation
- want our solution of the EDP also behaves like an outgoing wave!
- so let's look for a condition only \( u_\infty = A e^{ikr} \) satisfies.

\[
\left| \frac{\partial}{\partial r} u_\infty - i k u_\infty \right| = o\left( \frac{1}{r^2} \right) \text{ as } r \to \infty
\]

Sommerfeld Radiation Condition!
consider $-\Delta u - k^2 u = f$ in $\mathbb{R}^n$.

$u = G_k * f$ is a solution

.. that satisfies, putting $x = r\omega$

$$A(\omega) \frac{e^{ikr}}{r} + o(r^{-2}) \text{ as } r \to \infty$$

right choice of sign!
Steklov-Poincare Operator
Helmholtz Equation: From the Wave equation to Radiation condition
Uniqueness of the Exterior Dirichlet Problem
Boundary Integral Operators
Representation formula
Existence of the Exterior Dirichlet Problem
Uniqueness of the EDP!

**Theorem**

*There is at most one radiating solution* \( u \in H^1_{loc}(\Omega^c) \) *of*

\[
-\Delta u - k^2 u = 0 \quad \text{on} \quad \Omega^c
\]

\[
\gamma_D^c u = g \quad \text{on} \quad \Gamma,
\]

*where* \( g \in H^{\frac{1}{2}}(\Gamma) \).
Uniqueness

need the following Lemma established by Rellich:

Lemma

Let $k > 0$ and $u$ a solution of

$$-\Delta u - k^2 u = 0 \text{ on } B_{\rho_0}(0)^c$$

and suppose that

$$\lim_{\rho \to \infty} \int_{|x| = \rho} |u(x)|^2 d\sigma = 0$$

then $u = 0$ on $B_{\rho_0}(0)^c$. 
and, with a little work, it follows that

**Corollary**

Let \( u \in H^1_{loc}(\Omega^c) \) be a solution of the homogeneous exterior Helmholtz equation,

\[
-\Delta u - k^2 u = 0 \quad \text{on} \quad \Omega^c
\]

and satisfies in addition to the Sommerfeld radiation condition also

\[
\Im \left( k \int_{\Gamma} (\gamma_N u)(\gamma_D u) \, d\sigma \right) \geq 0,
\]

then \( u = 0 \) on \( \Omega^c \).
Uniqueness

Proof:

- Let's bound our domain: $\Omega_{\rho}^c = \Omega^c \cap B_{\rho}$

Green's 1st

$$\Rightarrow \int_{\Omega_{\rho}^c} \nabla \bar{u} \cdot \nabla u - \bar{k}^2 \bar{u} u \, dx = \langle (-\Delta - k^2) u, u \rangle_{L^2(\Omega_{\rho}^c)}$$

$$+ \langle \gamma_N u, \gamma_D u \rangle_{L^2(\partial \Omega_{\rho}^c)}$$

- first term vanishes by assumption. Multiply with $k$ and take the imaginary part

$$\Im(k) \int_{\Omega_{\rho}^c} |\nabla u|^2 - |k|^2 |u|^2 \, dx = \int_{\partial B_{\rho}} \Im \left( k \frac{\partial \bar{u}}{\partial \nu} u \right) \, d\sigma - \int_{\Gamma} \Im \left( k \frac{\partial \bar{u}}{\partial \nu} u \right) \, d\sigma$$
Uniqueness

- last integral can be rewritten since

\[
\left| \frac{\partial u}{\partial \rho} - iku \right|^2 = \left| \frac{\partial u}{\partial \rho} \right|^2 + |k|^2|u|^2 + 2\Im\left( k \frac{\partial \bar{u}}{\partial \nu} u \right)
\]

\[
\Rightarrow \Im(k) \int_{\Omega_{\rho}} |\nabla u|^2 + |k|^2|u|^2 \, dx + \frac{1}{2} \int_{\partial B_{\rho}} \left| \frac{\partial u}{\partial \rho} \right|^2 + |k|^2|u|^2 \, d\sigma =
\]

\[
+ \int_{\partial B_{\rho}} \left| \frac{\partial u}{\partial \rho} - iku \right|^2 \, d\sigma - \int_\Gamma \Im\left( k \frac{\partial \bar{u}}{\partial \nu} u \right) \, d\sigma.
\]

- \( \Im(k) > 0 \Rightarrow \int_{\Omega_{\rho}} |u|^2 \, dx \to 0 \)

- \( \Im(k) = 0 \Rightarrow \text{Rellich applies} \)
Outline

- Steklov-Poincare Operator
- Helmholtz Equation: From the Wave equation to Radiation condition
- Uniqueness of the Exterior Dirichlet Problem
- Boundary Integral Operators
- Representation formula
- Existence of the Exterior Dirichlet Problem
As for the Laplace, we define the boundary integral operators as

\[(V_k w)(x) = \int_{\Gamma} G_k(x, y) w(y) ds_y\]

\[(K_k v)(x) = \int_{\Gamma} \frac{\partial}{\partial n_y} G_k(x, y) v(y) ds_y\]

\[(K'_k v)(x) = \int_{\Gamma} \frac{\partial}{\partial n_x} G_k(x, y) v(y) ds_y\]

\[(W_k v)(x) = -\frac{\partial}{\partial n_x} \int_{\Gamma} \frac{\partial}{\partial n_y} G_k(x, y) v(y) ds_y\]
Integral operators and Boundary Integral Equations

- with the same properties as for the Laplace Theorem

For a bounded Lipschitz domain, the boundary integral operators

\[ V_k : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \]
\[ K_k : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \]
\[ K'_k : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \]
\[ W_k : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma) \]

are continuous.

Remark

One has the equalities \( W_k = K_k + \frac{1}{2} \text{Id} \) and \( V_k = K'_k + \frac{1}{2} \text{Id} \).
well... not exactly the same properties: $V_k$ no longer elliptic!

but we still have

**Lemma**

$V_k : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ is coercive, i.e. there exists a compact operator $C : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ such that the Gardings inequality

$$\langle V_k w, w \rangle_{L^2(\Gamma)} + \langle Cw, w \rangle_{L^2(\Gamma)} \geq \text{const} \ ||w||_{H^{-\frac{1}{2}}(\Gamma)}^2 \quad \forall w \in H^{-\frac{1}{2}}(\Gamma)$$

holds.
the single and double layer potential give us radiating solution:

**Theorem**

If \( \phi \in H^{-\frac{1}{2}}(\Gamma) \) then for \( u = \Psi_{SL}\phi \) satisfies the Sommerfeld radiation condition and

\[
(\Delta + k^2)u = 0 \text{ in } \mathbb{R}^n \setminus \Gamma
\]

Analogously for \( u = \Psi_{DL}\phi \) if \( \phi \in H^{\frac{1}{2}}(\Gamma) \).
suppose $u \in H^1_{loc}(\Omega^c)$ with $(-\Delta - k^2)u = 0$ in $\Omega \cup \Omega^c$ satisfies

$$u = -\Psi_{SL}[\gamma Nu] + \Psi_{DL}[\gamma Du] \text{ in } \Omega \cup \Omega^c$$

→ Integral representation formula

take traces to get the exterior Calderon projection

$$\begin{pmatrix} \gamma_D^c u \\ \gamma_N^c u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K_k & -V_k \\ -W_k & \frac{1}{2} I - K'_k \end{pmatrix} \begin{pmatrix} \gamma_D^c u \\ \gamma_N^c u \end{pmatrix}$$

different signs w.r.t to the interior Calderon
boundary integral equations to the exterior Dirichlet problem:

- Direct Method - integral equality of the first kind: Let $g \in H_{1/2}^1(\Gamma)$. Find $\psi \in H^{-1/2}(\Gamma)$ such that

$$\langle V_k \psi, \varphi \rangle_{L^2(\Gamma)} = -\frac{1}{2} \langle g, \varphi \rangle_{L^2(\Gamma)} + \langle K_k g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma)$$

- Direct Method - integral equality of the second kind: Let $g \in H_{1/2}^1(\Gamma)$. Find $\psi \in H^{-1/2}(\Gamma)$ such that

$$\frac{1}{2} \langle \psi, \varphi \rangle_{L^2(\Gamma)} + \langle K'_k \psi, \varphi \rangle_{L^2(\Gamma)} = -\langle W_k g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{1/2}(\Gamma)$$
Indirect Method - using the single layer potential: Let \( g \in H^{\frac{1}{2}}(\Gamma) \). Find \( \psi \in H^{-\frac{1}{2}}(\Gamma) \) such that

\[
\langle V_k \psi, \varphi \rangle_{L^2(\Gamma)} = \langle g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma)
\]

Indirect Method - using the double layer potential: Let \( g \in H^{\frac{1}{2}}(\Gamma) \). Find \( \psi \in H^{-\frac{1}{2}}(\Gamma) \) such that

\[
\frac{1}{2} \langle \psi, \varphi \rangle_{L^2(\Gamma)} + \langle K_k \psi, \varphi \rangle_{L^2(\Gamma)} = \langle g, \varphi \rangle_{L^2(\Gamma)} \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma)
\]
Steklov-Poincare Operator
Helmholtz Equation: From the Wave equation to Radiation condition
Uniqueness of the Exterior Dirichlet Problem
Boundary Integral Operators
Representation formula
Existence of the Exterior Dirichlet Problem
for the exterior Calderon operator we assumed Greens representation formula.. does it hold?

yes - thanks to the radiating property!

Theorem

Let \( g \in H^{\frac{1}{2}}(\Gamma) \). And suppose \( u \in H^{1}_{loc}(\Omega^c) \) is a radiating solution of

\[
-\Delta u - k^2 u = 0 \text{ on } \Omega^c
\]

\[
\gamma^c_D u = g \text{ on } \Gamma,
\]

then \( u \) has the integral representation

\[
u = \Psi_{DL}g - \Psi_{SL}(\gamma^c_N u).
\]
Proof idea:

- $\Omega^c_\rho = \Omega^c \cap B_\rho$

$$u(x) = +\Psi_{SL}(\gamma_N^c u)(x) - \Psi_{DL}(\gamma_D^c u)(x)$$

$$- \int_{\partial B_\rho} G(x, y) \gamma_N u(y) \, d\sigma + \int_{\partial B_\rho} \gamma_N G(x, y) \gamma_D u(y) \, d\sigma$$

- Sommerfeld radiation condition $\Rightarrow$ two last terms vanish as $\rho \to 0$
Outline

- Steklov-Poincare Operator
- Helmholtz Equation: From the Wave equation to Radiation condition
- Uniqueness of the Exterior Dirichlet Problem
- Boundary Integral Operators
- Representation formula
- Existence of the Exterior Dirichlet Problem
Existence

- we’re ready to prove the existence!
- ..by showing that we can solve the BIE.
- so we need

Theorem

Let \( g \in H^{\frac{1}{2}}(\Gamma) \). And suppose \( u \in H^1_{loc}(\Omega^c) \) is a radiating solution of

\[
-\Delta u - k^2 u = 0 \text{ on } \Omega^c \quad \gamma_D u = g \text{ on } \Gamma,
\]

then \( \phi = \gamma_N^c \in H^{-\frac{1}{2}}(\Gamma) \) is a solution of the boundary integral equation

\[
V_k \phi = (-\text{Id} / 2 + K)g \text{ on } \Gamma,
\]  

(1)

and \( u \) has the integral representation

\[
u = \Psi_{DL} g - \Psi_{SL} \phi.
\]  

(2)

Conversely, if \( \phi \in H^{-\frac{1}{2}}(\Gamma) \) is a solution (1), then formula (2) defines a solution \( u \in H^1_{loc}(\Omega^c) \) of the exterior Dirichlet problem.
Existence

- The wave number $k$ has influence on the BIE, namely

**Theorem**

$V_k$ is injective on $H^{-\frac{1}{2}}(\Gamma)$ iff $k^2$ is not an eigenvalue of $-\Delta$ of the interior Dirichlet problem, i.e.

$$-\Delta u = k^2 u \text{ on } \Omega, \quad \gamma_D u = 0 \Rightarrow u = 0 \text{ on } \Omega$$

The kernel of $V_k$ is given by

$$\ker(V_k) = \text{span} \left[ \gamma_N v : -\Delta v = k^2 v \text{ on } \Omega \text{ and } \gamma_D v = 0 \text{ on } \Gamma \right]$$
to finalize, we also need the Fredholm Alternative Theorem

Let $A \in B(X,Y)$ coercive. If $Au = 0$ only allows the trivial solution $u = 0$, then $Au = f$ is uniquely solvable for all $f \in Y$. Else $Au = f$ is solvable iff $\langle v, f \rangle = 0$ for all $v \in Y^* : A^* v = 0$. 
Existence

so at last

**Theorem**

Let $\Omega$ be a bounded Lipschitz domain with boundary $\Gamma$. Then for every $g \in H^{\frac{1}{2}}(\Gamma)$ the exterior Dirichlet problem

$$-\Delta u - k^2 u = 0 \text{ on } \Omega^c$$

$$\gamma_D^c u = g \text{ on } \Gamma$$

has a unique solution $u \in H^1_{loc}(\Omega^c)$ that satisfies the Sommerfeld radiation condition.
Existence

- need to solve $V_k \phi = (-\text{Id} / 2 + K_k)g$ on $\Gamma$
- $V_k$ coercive, so apply Fredholm!
- $k^2$ not an eigenvalue $\implies V_k$ invertible
- Else: need to check solvability condition
  $\langle w, (-\text{Id} / 2 + K_k)g \rangle_{L^2(\Gamma)} = 0$ for all $w \in \ker(V_k^*)$
- What’s $\ker(V_k^*)$? Can modify thm about $\ker(V_k)$!

$$\ker(V^*) = \text{span} \left[ \gamma_N v : -\Delta v = \bar{k}^2 v \text{ on } \Omega, \gamma_D v = 0 \text{ on } \Gamma \right]$$
so take such a $\gamma_N v$ and apply Green’s Identity

$$
\langle \gamma_N v, \left( -\frac{1}{2} I + K_k \right) g \rangle_{L^2(\Gamma)} = \langle \gamma_N v, \gamma_D (\Psi_{DL} g) \rangle_{L^2(\Gamma)} \\
= \langle \gamma_D v, \gamma_N (\Psi_{DL} g) \rangle_{L^2(\Gamma)} \\
- \langle (\Delta - \bar{k}^2) v, \Psi_{DL} g \rangle_{L^2(\Omega)} \\
+ \langle v, (\Delta - k^2) \Psi_{DL} g \rangle_{L^2(\Omega)} \\
= 0
$$

...since $\gamma_D v = 0$ on $\Gamma$, $(-\Delta - \bar{k}^2)v = 0$ and $(-\Delta - k^2)\Psi_{DL} g = 0$ on $\Omega$. 