p-adic Analysis Compared to Real
Lecture 1

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1 Normed Fields & non-Archimedean Norms

Definition 1.1. A *metric* on a non-empty set X is a function

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying the following properties:

(1) $d(x, y) = 0 \iff x = y$

(2) $d(x, y) = d(y, x) \quad \forall x, y \in X$

(3) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$

(X, d) is called a *metric space.*

Definition 1.2. A sequence (x_n) in a metric space (X, d) is called *Cauchy sequence* if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon \; \forall m, n > N$.
(X, d) is a *complete* metric space if any Cauchy sequence in X has a limit in X.

Definition 1.3. Let F be a field. A *norm* on F is a map $\| \cdot \| : F \to \mathbb{R}_{\geq 0}$ that satisfies:

(1) $\|x\| = 0 \iff x = 0$

(2) $\|xy\| = \|x\|\|y\| \quad \forall x, y \in F$

(3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in F$ (*triangle inequality*)

$(F, \| \cdot \|)$ is called a *normed field.*
A norm is *trivial* if $\|0\| = 0$ and $\|x\| = 1 \; \forall x \in F \backslash \{0\}$.

Remark 1.1. A norm $\|\cdot\|$ on a field F induces a metric on F by:

$$F \times F \longrightarrow \mathbb{R}_{\geq 0} \,:\, (x,y) \longmapsto \|x-y\|$$

This allows us to regard a normed field $(F, \|\cdot\|)$ as a metric space.

Proposition 1.1. *For any $x, y \in F$ we have:*

(a) $\|1\| = \|-1\| = 1$

(b) $\|x\| = \|-x\|$

(c) $\|x \pm y\| \geq |\,\|x\| - \|y\|\,|$

(d) $\|x - y\| \leq \|x\| + \|y\|$

(e) $\|x/y\| = \|x\|/\|y\|$

(f) $\|n\| \leq n \;\forall n \in \mathbb{N}$ *(on the left hand side: $n := n \cdot 1_F \in F$)*

Proof. (a) $\|1\| = \|(\pm 1)\cdot(\pm 1)\| = \|\pm 1\|^2 \implies \|\pm 1\| = 1$

(b) $\|-x\| = \|-1\|\|x\| = \|x\|$

(c) $\|x\| = \|x \pm y \mp y\| \leq \|x \pm y\| + \|y\| \implies \|x\| - \|y\| \leq \|x \pm y\|$
$\|y\| = \|y \pm x \mp x\| \leq \|y \pm x\| + \|x\| \implies \|y\| - \|x\| \leq \|y \pm x\|$
Thus $\|x \pm y\| \geq |\,\|x\| - \|y\|\,|$.

(d) Follows from (b) and the triangle inequality.

(e) $\|y\|\|x/y\| = \|x\|$

(f) Follows by induction from (a) and the triangle inequality.

□

Definition 1.4. A norm is called *non-Archimedean* if it satisfies the *strong triangle inequality*:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}.$$

If a norm does not satisfy the strong triangle inequality it is said to be *Archimedean*.

Remark 1.2. The strong triangle inequality clearly implies the triangle inequality.

We call a metric that is induced by a non-Archimedean norm an *ultra-metric* and the corresponding metric space an *ultra-metric space*.
An ultra-metric satisfies the strong triangle inequality:

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

Proposition 1.2. $\|\cdot\|$ *is non-Archimedean* $\iff$ $\|n\| \leq 1\, \forall n \in \mathbb{Z}$

Proof. " $\Longrightarrow$ ": Induction: $\|1\| = 1 \leq 1$. Suppose that $\|k\| \leq 1\, \forall k \in \{1, \ldots, n-1\}$. $\|n\| = \|n-1+1\| \leq \max\{\|n-1\|, \|1\|\} = 1$. Hence $\|n\| \leq 1, \forall n \in \mathbb{N}$. Since we have that $\|n\| = \|-n\|$, the result follows.
"$\Longleftarrow$":

$$\begin{aligned}
\|x+y\|^n = \|(x+y)^n\| &= \left\| \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right\| \\
&\leq \sum_{k=0}^{n} \left\| \binom{n}{k} \right\| \|x\|^k \|y\|^{n-k} \\
&\leq \sum_{k=0}^{n} \|x\|^k \|y\|^{n-k} \\
&\leq (n+1)[\max\{\|x\|, \|y\|\}]^n
\end{aligned}$$

Hence $\|x+y\| \leq (n+1)^{1/n} \max\{\|x\|, \|y\|\}$.
Letting n tend to ∞ we get:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}.$$

□

Proposition 1.3. *The following are equivalent:*

(i) $\|\cdot\|$ *is Archimedean*

(ii) $\|\cdot\|$ *has the* Archimedean property*:*
given $x, y \in F, x \neq 0\ \exists n \in \mathbb{N}$ *such that* $\|nx\| > \|y\|$

(iii) $\sup\{\|n\| : n \in \mathbb{Z}\} = +\infty$

Proof. "(i) $\Longrightarrow$ (iii)": By Proposition 1.2 $\exists n \in \mathbb{Z}$ such that $\|n\| > 1$
$\Longrightarrow \|n^k\| = \|n\|^k \stackrel{k\to\infty}{\longrightarrow} \infty$.
"(iii) $\Longrightarrow$ (ii)": Given $x, y \in F$, $x \neq 0$ we can choose $n \in \mathbb{N}$ such that:

$$\|n\| > \frac{\|y\|}{\|x\|} \implies \|nx\| > \|y\|.$$

"(ii) $\Longrightarrow$ (i)": Take $x, y \in F$, $x \neq 0$ such that $\|x\| \leq \|y\|$. By (ii) $\exists n \in \mathbb{N}$ such that $\|n\| > \frac{\|y\|}{\|x\|} \geq 1$, hence the result follows from Proposition 1.2. □

Proposition 1.4. *If $\|\cdot\|$ is non-Archimedean we have:*

$$\|x-a\| < \|a\| \implies \|a\| = \|x\|$$

Proof.

$$\|x\| = \|x-a+a\| \leq \max\{\|x-a\|, \|a\|\} = \|a\|$$

and

$$\|a\| = \|a-x+x\| \leq \max\{\|a-x\|, \|x\|\}.$$

Suppose that $\|a-x\| > \|x\|$, then $\|x-a\| \geq \|a\|$ which contradicts the assumption. Therefore we get that $\|a\| \leq \|x\|$ and hence $\|a\| = \|x\|$. □

Proposition 1.5. *Any triangle in an ultra-metric space (X, d) is* isosceles *and the length of its base does not exceed the length of the other two sides.*

Proof. Let $x, y, z \in F$. W.l.o.g. assume that $d(x, y) < d(x, z)$. Then:

$$d(y, z) \leq \max\{d(x, y), d(x, z)\} = d(x, z)$$

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(y, z)$$

Therefore: $d(x, z) = d(y, z)$. □

Proposition 1.6. *Let $\|\cdot\|$ be non-Archimedean. Any point of an open ball $B(a, r) := \{x \in F : \|x-a\| < r\}$ is its center. The same is true for closed balls.*

Proof. Fix any $b \in B(a, r)$.
Choose $x \in B(a, r)$, then

$$\|x-b\| = \|x-a+a-b\| \leq \max\{\|x-a\|, \|a-b\|\} < r.$$

Therefore we get that $B(a, r) \subset B(b, r)$. Similarly, we get that $B(b, r) \subset B(a, r)$ and thus $B(a, r) = B(b, r)$. This argument can easily be adapted to the case of closed balls. □

2 The Completion of a Normed Field

It is well-known how the real numbers can be constructed from the rationals as equivalence classes of Cauchy sequences. In this section we will generalize this construction to arbitrary normed fields.

Let $(F, \|\cdot\|)$ be a normed field. Let CF denote the set of all Cauchy sequences in F. Componentwise addition and multiplication turns CF into a commutative ring which obviously contains lots of zero divisors and hence is in no way a field.

However, we will get a field from CF by identifying sequences which should have the same limit.

First we embed F into CF via the map

$$F \to CF: \quad a \mapsto \hat{a} := (a, a, a, ...).$$

Note that $\hat{0}$ and $\hat{1}$ are the neutral elements of addition and multiplication in CF.

Let

$$N := \{(a_n) \in F^{\mathbb{N}} | \lim_{n \to \infty} \|a_n\| = 0\}$$

be the set of all null sequences in F. Note that $N \subset CF$ since every converging sequence is a Cauchy sequence.

Proposition 2.1. *This N is a maximal ideal in CF.*

Proof. *Claim 1:* $(N,+)$ is a group.

Let $(a_n), (b_n)$ be null sequences. Let $\varepsilon > 0$ and $N, M \in \mathbb{N}$ such that $\forall n > N : \|a_n\| < \varepsilon/2$ and $\forall m > M : \|b_m\| < \varepsilon/2$. Then, by the triangle inequality, for all $k > \max\{N, M\} : \|a_k + b_k\| \leq \|a_k\| + \|b_k\| < \varepsilon$. That the additive inverse of a null sequence is again a null sequence is the direct consequence of Proposition 1.1 and that $\hat{0}$ is a null sequence is obvious.

Claim 2: Let (a_n) be a null sequence and (x_n) be any Cauchy sequence. Then, $(a_n x_n)$ is a null sequence.

First we prove that every Cauchy sequence is bounded. Let $C := \sup\{\|x_n\| | n \in \mathbb{N}\}$. Since for big enough n we know that $\|x_n\|$ can only lie in some interval $[\|x_m\| - \varepsilon, \|x_m\| + \varepsilon]$ for an appropriate m, we get $C < \infty$.

Now we prove that the product of a bounded sequence with a nullsequence is a nullsequence. Let $\varepsilon > 0$. Let now $N \in \mathbb{N}$ be such that $\forall n > N : \|a_n\| < \varepsilon/C$. Then, for $n > N : \|a_n x_n\| < \varepsilon/C \cdot C = \varepsilon$.

Hence, N is an ideal.

Claim 3: N is maximal.

Let (a_n) be a Cauchy sequence which is not a null sequence. We want to prove that $\hat{1} \in (N, (a_n))$. Since (a_n) is not a null sequence, there exists a $c > 0$ and an $N > 0$ such that for all $n > N : \|a_n\| > c$. Since for any $\varepsilon > 0$ and big enough n, m we have

$$\left\| \frac{1}{a_n} - \frac{1}{a_m} \right\| = \left\| \frac{a_m - a_n}{a_m a_n} \right\| \leq \frac{\varepsilon}{c^2}.$$

Define now

$$b_n := \begin{cases} 1 & \text{if } a_n = 0 \\ 1/a_n & \text{else.} \end{cases}$$

By the above, (b_n) is a Cauchy sequence. Define a null sequence (x_n) as follows

$$x_n := \begin{cases} 1 & \text{if } a_n = 0 \\ 0 & \text{else.} \end{cases}$$

This is a null sequence since for big enough n, all x_n are zero. Now,

$$1 = a_n b_n + x_n$$

for all $n \in \mathbb{N}$ and hence $\hat{1} \in (N, (a_n))$. □

Define

$$\hat{F} := CF/N.$$

Since N is maximal, $\hat{F}$ is a field. We now extend the norm of F to a norm on $\hat{F}$.

Definition 2.1. Let $a \in \hat{F}$. Then, the *norm of a* is defined by

$$\|a\| := \lim_{n\to\infty} \|a_n\|,$$

where $a = (a_n) + N$.

Proposition 2.2. *This is a well-defined norm on $\hat{F}$.*

Proof. If two Cauchy sequences $(a_n), (b_n)$ differ only by a null-sequence, then by Proposition 1.1 we immediately get that

$$\lim_{n\to\infty} \|a_n\| = \lim_{n\to\infty} \|b_n\|$$

and hence $\|\cdot\|$ is well-defined.

From the definition it directly follows that the elements with norm zero are exactly the null sequences. Multiplicativity and the triangle inequality also follow immediately. □

Definition 2.2. The normed field $(\hat{F}, \|\cdot\|)$ is called the *completion of F with respect to the norm $\|\cdot\|$*.

This terminology is justified by the following theorem.

Theorem 2.3. *The normed field $(\hat{F}, \|\cdot\|)$ is complete and F is dense in $\hat{F}$.*

Proof. First we prove the second statement. Let (a_n) be a Cauchy sequence in F. Then, $(\hat{a}_n)$ is a sequence of (constant) Cauchy sequences and we have

$$\lim_{n,m\to\infty} \|a_n - a_m\| = 0.$$

Hence, $(\hat{a}_n)$ converges to the class represented by (a_n). Hence, F is dense in $\hat{F}$.

Now let (A_n) be a Cauchy sequence of Cauchy sequences in F, hence a representative of a Cauchy sequence in $\hat{F}$. Since F is dense and A_n is a Cauchy sequence for every n, we know that there exists a Cauchy sequence $(\hat{a}_n)$ for all n such that

$$A_n - (\hat{a}_n) < \frac{1}{n}.$$

It follows that

$$(a_n) - (A_n) = ((a_n) - (\hat{a}_n)) - (A_n - (\hat{a}_n))$$

is a null sequence in $\hat{F}$. Therefore,

$$\lim_{n\to\infty} \|(a_n) - A_n\| = 0$$

. □

The only thing left to check is that the field operations of $\hat{F}$ come from F in a continuous way.

Proposition 2.4. *Let (a_n) and (b_n) be Cauchy sequences in $F \subset \hat{F}$. Then,*

$$\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$$
$$\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$$

Proof. From the proof of the above theorem follows

$$\lim_{n\to\infty}(a_n + b_n) = (a_n + b_n) = (a_n) + (b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$$

where we denote the class represented by (a_n) also by (a_n). Analog for multiplication. □

3 The field of p-adic numbers $\mathbb{Q}_p$

Definition 3.1. Let $p \in \mathbb{N}$ prime. For $0 \neq x \in \mathbb{Q}$ we define the p-adic order of x

$$ord_p(x) = \begin{cases} max\{n \in \mathbb{N} | p^n \text{ divides } x\}, \text{ if } x \in \mathbb{Z} \\ ord_p(a) - ord_p(b), \text{ if } x = a/b, b \neq 0, a, b \in \mathbb{Z} \end{cases}$$

Remark 3.1. For $x = a/b, y = c/d \in \mathbb{Q}$ we have

$$\begin{aligned} ord_p(\frac{a}{b}\frac{c}{d}) &= ord_p(ac) - ord_p(bd) \quad (1) \\ &= ord_p(a) + ord_p(c) - ord_p(b) - ord_p(d) \\ &= ord_p(\frac{a}{b}) + ord_p(\frac{c}{d}) \end{aligned}$$

and if $x \neq -y$

$$\begin{aligned} & ord_p(\frac{a}{b}+\frac{c}{d}) = ord_p(\frac{ad+cb}{bd}) = ord_p(ad+cb) - ord_p(bd) \\ & \geq min\{ord_p(ad), ord_p(cb)\} - ord_p(b) - ord_p(d) \\ & = min\{ord_p(a)+ord_p(d), ord_p(c)+ord_p(b)\} - ord_p(b) - ord_p(d) \\ & = min\{ord_p(a)-ord_p(b), ord_p(c)-ord_p(d)\} \\ & = min\{ord_p(\frac{a}{b}), ord_p(\frac{c}{d})\} \end{aligned} \tag{2}$$

Definition 3.2. On $\mathbb{Q}$ we define the p-adic norm

$$|x|_p = \begin{cases} p^{-ord_p(x)}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases}$$

Proposition 3.1. $|.|_p$ *is a non-Archimedean norm on* $\mathbb{Q}$.

Proof. $|x|_p = 0 \iff x = 0$ follows from the definition of the $|.|_p$ norm.
$|xy|_p = |x|_p|y|_p$ follows from (1).
For the strong triangle inequality we have

$$|x+y|_p = p^{-ord_p(x+y)} \leq max\{p^{-ord_p(x)}, p^{-ord_p(y)}\} = max\{|x|_p, |y|_p\}$$

from (2). □

Remark 3.2. Unlike the euclidean norm on $\mathbb{Q}$, given two numbers $a, b \in \mathbb{Q}$ with $|a|_p < |b|_p$ we can't always find a third number $c \in \mathbb{Q}$ so that $|a|_p < |c|_p < |b|_p$. In particular, $|.|_p$ only takes values in $\{p^k | k \in \mathbb{Z}\} \cup \{0\}$.

Definition 3.3. Let $p \in \mathbb{N}$ be prime. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to $|.|_p$, and its elements are equivalence classes of Cauchy sequences.
For an element $a \in \mathbb{Q}_p$ and a Cauchy sequence (a_n) representing a, we defined the norm of a as

$$|a|_p = \lim_{n\to\infty} |a_n|_p$$

Remark 3.3. By remark 3.2, the norm of $a \in \mathbb{Q}_p$ only takes values in $\{p^k | k \in \mathbb{Z}\} \cup \{0\}$, just like $|.|_p$. Also, if $|a|_p = p^k \neq 0$, then for any Cauchy sequence representing a there is an N so that $|a_n|_p = p^k$ for $n > N$.

How does an element of $\mathbb{Q}_p$ look like? The following proposition will give a way to construct one of them.

Proposition 3.2. *Let $d_{-m} \neq 0$ and $0 \leq d_i < p$ integers. Then the partial sums of the series*

$$a = d_{-m}p^{-m} + d_{-m+1}p^{-m+1} + \dots + d_{-1}p^{-1} + d_0 + d_1 p + d_2 p^2 + \dots \tag{3}$$

form a Cauchy sequence and therefore a is an element of $\mathbb{Q}_p$.

Proof. Let $\epsilon > 0$. Then we can find $N \in \mathbb{N}$ so that $p^{-N} < \epsilon$, and for $n, k > N$, WLOG $k > n$, we have

$$\left| \sum_{i=-m}^{k} d_i p^i - \sum_{i=-m}^{n} d_i p^i \right|_p = \left| \sum_{i=n+1}^{k} d_i p^i \right|_p \leq max\{|d_{n+1}p^{n+1}|_p, \dots, |d_k p^k|_p\} \leq p^{-N} < \epsilon$$

□

We might now wonder if every element of $\mathbb{Q}_p$ looks like a series (3).
The following propositions will help us prove that every $a \in \mathbb{Q}_p$ actually has a *unique* Cauchy sequence representing it that looks like (3).

Proposition 3.3. *Let $x \in \mathbb{Q}$ with $|x|_p \leq 1$. Then for any i there is a unique integer $\alpha \in \{0, 1, \dots, p^i - 1\}$ so that $|x - \alpha|_p \leq p^{-i}$.*

Proof. Let $x = a/b$ with a and b relatively prime. Since $|x|_p = p^{-ord_p(a)+ord_p(b)} \leq 1$ we get $ord_p(b) = 0$, that is, b and p^i are relatively prime for any i. We can then find integers m and n so that $np^i + mb = 1$. For $\alpha = am$ we get

$$|\alpha - x|_p = \left| am - \frac{a}{b} \right|_p = \left| \frac{a}{b} \right|_p |mb - 1|_p \leq |mb - 1|_p = |np^i|_p = |n|_p p^{-i} \leq p^{-i}$$

There is exactly a multiple cp^i of p^i so that $cp^i + \alpha \in \{0, 1, \dots, p^i - 1\}$, and we have

$$|cp^i + \alpha - x|_p \leq max\{|\alpha - x|_p, |cp^i|\} \leq max\{p^{-i}, p^{-i}\} = p^{-i}$$

□

Theorem 3.4. *Let $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$. Then there is exactly one Cauchy sequence (a_n) representing a so that for any i*
i) $0 \leq a_i < p^i$
ii) $a_i \equiv a_{i+1}$ (mod p^i)

Proof. Let (c_n) be a Cauchy sequence representing a. Since $|c_n|_p \to |a|_p \leq 1$, there is an N so that $|c_n|_p \leq 1$ for any $n > N$. (If $|a|_p = 1$ this still holds because of remark 3.3)
By replacing the first N elements we can find an equivalent Cauchy sequence so

that $|b_n|_p \leq 1$ for any n. Now, for every $j = 1, 2, \ldots$ let $N(j)$ be so that $N(j) \geq j$ and

$$|b_i - b_{i'}|_p \leq p^{-j} \ \forall i, i' \geq N(j)$$

From the previous proposition we know that for any j we can find integers $0 \leq a_j < p^j$ (condition i)) so that

$$|a_j - b_{N(j)}|_p \leq p^{-j}$$

These a_j also satisfy condition ii):

$$\begin{aligned}
|a_{j+1} - a_j|_p &= |a_{j+1} - b_{N(j+1)} + b_{N(j+1)} - b_{N(j)} + b_{N(j)} - a_j|_p \\
&\leq max\{|a_{j+1} - b_{N(j+1)}|_p, |b_{N(j+1)} - b_{N(j)}|_p, |b_{N(j)} - a_j|_p\} \\
&\leq max\{p^{-j-1}, p^{-j}, p^{-j}\} = p^{-j}
\end{aligned}$$

This sequence is equivalent to (b_n): for any j take $i \geq N(j)$

$$\begin{aligned}
|a_i - b_i|_p &= |a_i - a_j + a_j - b_{N(j)} + b_{N(j)} - b_i|_p \\
&\leq max\{|a_i - a_j|_p, |a_j - b_{N(j)}|_p, |b_{N(j)} - b_i|_p\} \\
&\leq max\{p^{-j}, p^{-j}, p^{-j}\} = p^{-j}
\end{aligned}$$

so $|a_i - b_i| \to 0$.

Now, to show uniqueness, let (d_n) be another Cauchy sequence satisfying conditions i) and ii) and let $(a_n) \neq (d_n)$, that is, for some i_0, $a_{i_0} \neq d_{i_0}$. Since a_{i_0} and d_{i_0} are between 0 and p^{i_0}, $a_{i_0} \neq d_{i_0} \pmod{p^{i_0}}$. From condition ii) we have that for $i > i_0$, $a_i = a_{i_0} \neq d_{i_0} = d_i \pmod{p^{i_0}}$, that is, $a_i \neq d_i \pmod{p^{i_0}}$ and therefore

$$|a_i - d_i|_p > p^{-i_0}$$

doesn't converge to 0 and (a_n) and (d_n) aren't equivalent. □

Remark 3.4. For $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$ we can write the Cauchy sequence (a_n) representing a from the previous proposition as

$$a_i = d_0 + d_1 p + \cdots + d_{i-1} p^{i-1}$$

for $d_i \in \{0, 1, \ldots, p-1\}$ and a is represented by the convergent series

$$a = \sum_{i=0}^{\infty} d_i p^i$$

which we can think of as a number written in base p which keeps extending to the left

$$a = \ldots d_n \ldots d_1 d_0$$

Remark 3.5. If $a \in \mathbb{Q}_p$ with $|a|_p = p^m > 1$ then $a' = ap^m$ satisfies $|a'| = p^m p^{-m} = 1$ and we can then write

$$a = a'p^{-m} = p^{-m} \sum_{i=0}^{\infty} c_i p^i = \sum_{i=-m}^{\infty} d_i p^i$$

with $d_{-m} = c_0 \neq 0$ and a becomes a fraction in base p with finitely many digits after the point and which extends infinitely to the left

$$a = \ldots d_n \ldots d_1 d_0 . d_{-1} d_{-2} \ldots d_{-m}$$

Definition 3.4. This way of writing $a \in \mathbb{Q}_p$ as a number written in base p which keeps extending to the left is called the *p-adic expansion of a*. This will either look like

$$a = \ldots d_n \ldots d_1 d_0$$

for $d_i \in \{0, 1, \ldots, p-1\}$, if $|a|_p \leq 1$, or like

$$a = \ldots d_n \ldots d_1 d_0 . d_{-1} d_{-2} \ldots d_{-m}$$

for $d_i \in \{0, 1, \ldots, p-1\}$ and $d_{-m} \neq 0$, if $|a|_p > 1$.