1 Normed Fields & non-Archimedean Norms

Definition 1.1. A metric on a non-empty set $X$ is a function

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$$

satisfying the following properties:

(1) $d(x, y) = 0 \iff x = y$

(2) $d(x, y) = d(y, x) \quad \forall x, y \in X$

(3) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$

$(X, d)$ is called a metric space.

Definition 1.2. A sequence $(x_n)$ in a metric space $(X, d)$ is called Cauchy sequence if $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon \forall m, n > N$.

$(X, d)$ is a complete metric space if any Cauchy sequence in $X$ has a limit in $X$.

Definition 1.3. Let $F$ be a field. A norm on $F$ is a map $\| \cdot \| : F \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

(1) $\|x\| = 0 \iff x = 0$

(2) $\|xy\| = \|x\| \|y\| \quad \forall x, y \in F$

(3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in F$ (triangle inequality)

$(F, \| \cdot \|)$ is called a normed field.

A norm is trivial if $\|0\| = 0$ and $\|x\| = 1 \forall x \in F \setminus \{0\}$. 
Remark 1.1. A norm $\| \cdot \|$ on a field $F$ induces a metric on $F$ by:

$$F \times F \rightarrow \mathbb{R}_{\geq 0} : (x, y) \mapsto \| x - y \|$$

This allows us to regard a normed field $(F, \| \cdot \|)$ as a metric space.

Proposition 1.1. For any $x, y \in F$ we have:

(a) $\| 1 \| = \| - 1 \| = 1$

(b) $\| x \| = \| - x \|$

(c) $\| x \pm y \| \geq \| x \| - \| y \|$

(d) $\| x - y \| \leq \| x \| + \| y \|$

(e) $\| x/y \| = \| x \|/\| y \|$

(f) $\| n \| \leq n \forall n \in \mathbb{N}$ (on the left hand side: $n := n \cdot 1_F \in F$)

Proof. (a) $\| 1 \| = \|(\pm 1) \cdot (\pm 1)\| = \| \pm 1 \|^2 \Rightarrow \| \pm 1 \| = 1$

(b) $\| - x \| = \| - 1\|\| x \| = \| x \|$

(c) $\| x \| = \| x \pm y \| \leq \| x \pm y \| + \| y \| \quad \Rightarrow \quad \| x \| - \| y \| \leq \| x \pm y \|$

Thus $\| x \pm y \| \geq \| x \| - \| y \|$.  

(d) Follows from (b) and the triangle inequality.

(e) $\| y \| \| x/y \| = \| x \|$

(f) Follows by induction from (a) and the triangle inequality.

Definition 1.4. A norm is called non-Archimedean if it satisfies the strong triangle inequality:

$$\| x + y \| \leq \max\{ \| x \|, \| y \| \}.$$

If a norm does not satisfy the strong triangle inequality it is said to be Archimedean.

Remark 1.2. The strong triangle inequality clearly implies the triangle inequality.

We call a metric that is induced by a non-Archimedean norm an ultra-metric and the corresponding metric space an ultra-metric space. 

An ultra-metric satisfies the strong triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$
Proposition 1.2. $\| \cdot \|$ is non-Archimedean $\iff \|n\| \leq 1 \forall n \in \mathbb{Z}$

Proof. "$\implies$": Induction: $\|1\| = 1 \leq 1$. Suppose that $\|k\| \leq 1 \forall k \in \{1, \ldots, n-1\}$. $\|n\| = \|n - 1 + 1\| \leq \max\{\|n - 1\|, \|1\|\} = 1$. Hence $\|n\| \leq 1, \forall n \in \mathbb{N}$. Since we have that $\|n\| = \| - n\|$, the result follows.

"$\impliedby$":

\[ \|x + y\|^n = \|(x + y)^n\| = \left\| \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right\| \]

\[ \leq \sum_{k=0}^{n} \binom{n}{k} \|x\|^k \|y\|^{n-k} \]

\[ \leq \sum_{k=0}^{n} \|x\|^k \|y\|^{n-k} \]

\[ \leq (n + 1) \max\{\|x\|, \|y\|\} \]

Hence $\|x + y\| \leq (n + 1)^{1/n} \max\{\|x\|, \|y\|\}$.

Letting $n$ tend to $\infty$ we get:

\[ \|x + y\| \leq \max\{\|x\|, \|y\|\}. \]

\[ \square \]

Proposition 1.3. The following are equivalent:

(i) $\| \cdot \|$ is Archimedean

(ii) $\| \cdot \|$ has the Archimedean property:

\[ \text{given } x, y \in F, x \neq 0 \exists n \in \mathbb{N} \text{ such that } \|nx\| > \|y\| \]

(iii) $\sup\{\|n\| : n \in \mathbb{Z}\} = +\infty$

Proof. "(i) $\implies$ (iii)" : By Proposition 1.2 $\exists n \in \mathbb{Z}$ such that $\|n\| > 1$ $\implies \|n^k\| = \|n\|^k \stackrel{k \to \infty}{\to} \infty$.

"(iii) $\implies$ (ii)" : Given $x, y \in F, x \neq 0$ we can choose $n \in \mathbb{N}$ such that:

\[ \|n\| > \frac{\|y\|}{\|x\|} \implies \|nx\| > \|y\|. \]

"(ii) $\implies$ (i)" : Take $x, y \in F, x \neq 0$ such that $\|x\| \leq \|y\|$. By (ii) $\exists n \in \mathbb{N}$ such that $\|n\| > \frac{\|y\|}{\|x\|} \geq 1$, hence the result follows from Proposition 1.2. \[ \square \]
**Proposition 1.4.** If $\| \cdot \|$ is non-Archimedean we have:

$$\|x - a\| < \|a\| \implies \|a\| = \|x\|$$

*Proof.*

$$\|x\| = \|x - a + a\| \leq \max\{\|x - a\|, \|a\|\} = \|a\|$$

and

$$\|a\| = \|a - x + x\| \leq \max\{\|a - x\|, \|x\|\}.$$  

Suppose that $\|a - x\| > \|x\|$, then $\|x - a\| \geq \|a\|$ which contradicts the assumption. Therefore we get that $\|a\| \leq \|x\|$ and hence $\|a\| = \|x\|$.  

**Proposition 1.5.** Any triangle in an ultra-metric space $(X, d)$ is isosceles and the length of its base does not exceed the length of the other two sides.

*Proof.* Let $x, y, z \in F$. W.l.o.g. assume that $d(x, y) < d(x, z)$. Then:

$$d(y, z) \leq \max\{d(x, y), d(x, z)\} = d(x, z)$$

$$d(x, z) \leq \max\{d(x, y), d(y, z)\} = d(y, z)$$

Therefore: $d(x, z) = d(y, z)$.  

**Proposition 1.6.** Let $\| \cdot \|$ be non-Archimedean. Any point of an open ball $B(a, r) := \{x \in F : \|x - a\| < r\}$ is its center. The same is true for closed balls.

*Proof.* Fix any $b \in B(a, r)$. Choose $x \in B(a, r)$, then

$$\|x - b\| = \|x - a + a - b\| \leq \max\{\|x - a\|, \|a - b\|\} < r.$$  

Therefore we get that $B(a, r) \subset B(b, r)$. Similarly, we get that $B(b, r) \subset B(a, r)$ and thus $B(a, r) = B(b, r)$. This argument can easily be adapted to the case of closed balls.

2 The Completion of a Normed Field

It is well-known how the real numbers can be constructed from the rationals as equivalence classes of Cauchy sequences. In this section we will generalize this construction to arbitrary normed fields.  

Let $(F, \| \cdot \|)$ be a normed field. Let $CF$ denote the set of all Cauchy sequences in $F$. Componentwise addition and multiplication turns $CF$ into a commutative ring which obviously contains lots of zero divisors and hence is in no way a field.
However, we will get a field from $CF$ by identifying sequences which should have the same limit.

First we embed $F$ into $CF$ via the map

$$F \to CF : \quad a \mapsto \hat{a} := (a, a, a, \ldots).$$

Note that $\hat{0}$ and $\hat{1}$ are the neutral elements of addition and multiplication in $CF$.

Let

$$N := \{ (a_n) \in F^\mathbb{N} \mid \lim_{n \to \infty} \|a_n\| = 0 \}$$

be the set of all null sequences in $F$. Note that $N \subset CF$ since every converging sequence is a Cauchy sequence.

**Proposition 2.1.** This $N$ is a maximal ideal in $CF$.

**Proof.** Claim 1: $(N, +)$ is a group.

Let $(a_n), (b_n)$ be null sequences. Let $\varepsilon > 0$ and $N, M \in \mathbb{N}$ such that $\forall n > N : \|a_n\| < \varepsilon/2$ and $\forall m > M : \|b_m\| < \varepsilon/2$. Then, by the triangle inequality, for all $k > \max\{N, M\} : \|a_k + b_k\| \leq \|a_k\| + \|b_k\| < \varepsilon$. That the additive inverse of a null sequence is again a null sequence is the direct consequence of Proposition 1.1 and that $\hat{0}$ is a null sequence is obvious.

Claim 2: Let $(a_n)$ be a null sequence and $(x_n)$ be any Cauchy sequence. Then, $(a_n x_n)$ is a null sequence.

First we prove that every Cauchy sequence is bounded. Let $C := \sup \{\|x_n\| \mid n \in \mathbb{N}\}$. Since for big enough $n$ we know that $\|x_n\|$ can only lie in some interval $[\|x_m\| - \varepsilon, \|x_m\| + \varepsilon]$ for an appropriate $m$, we get $C < \infty$.

Now we prove that the product of a bounded sequence with a nullsequence is a nullsequence. Let $\varepsilon > 0$. Let now $N \in \mathbb{N}$ be such that $\forall n > N : \|a_n\| < \varepsilon/C$. Then, for $n > N : \|a_n x_n\| < \varepsilon/C \cdot C = \varepsilon$.

Hence, $N$ is an ideal.

Claim 3: $N$ is maximal.

Let $(a_n)$ be a Cauchy sequence which is not a null sequence. We want to prove that $\hat{1} \in (N, (a_n))$. Since $(a_n)$ is not a null sequence, there exists a $c > 0$ and an $N > 0$ such that for all $n > N : \|a_n\| > c$. Since for any $\varepsilon > 0$ and big enough $n, m$ we have

$$\left\| \frac{1}{a_n} - \frac{1}{a_m} \right\| = \left\| \frac{a_m - a_n}{a_m a_n} \right\| \leq \frac{\varepsilon}{c^2},$$

Define now

$$b_n := \begin{cases} 1 & \text{if } a_n = 0 \\ 1/a_n & \text{else.} \end{cases}$$
By the above, \((b_n)\) is a Cauchy sequence. Define a null sequence \((x_n)\) as follows
\[
x_n := \begin{cases} 
1 & \text{if } a_n = 0 \\
0 & \text{else.}
\end{cases}
\]
This is a null sequence since for big enough \(n\), all \(x_n\) are zero. Now,
\[
1 = a_n b_n + x_n
\]
for all \(n \in \mathbb{N}\) and hence \(\hat{1} \in (N, (a_n))\).

Define
\[
\hat{F} := CF/N.
\]
Since \(N\) is maximal, \(\hat{F}\) is a field. We now extend the norm of \(F\) to a norm on \(\hat{F}\).

**Definition 2.1.** Let \(a \in \hat{F}\). Then, the norm of \(a\) is defined by
\[
\|a\| := \lim_{n \to \infty} \|a_n\|
\]
where \(a = (a_n) + N\).

**Proposition 2.2.** This is a well-defined norm on \(\hat{F}\).

**Proof.** If two Cauchy sequences \((a_n), (b_n)\) differ only by a null-sequence, then by Proposition 1.1 we immediately get that
\[
\lim_{n \to \infty} \|a_n\| = \lim_{n \to \infty} \|b_n\|
\]
and hence \(\| \cdot \|\) is well-defined.

From the definition it directly follows that the elements with norm zero are exactly the null sequences. Multiplicativity and the triangle inequality also follow immediately.

**Definition 2.2.** The normed field \((\hat{F}, \| \cdot \|)\) is called the completion of \(F\) with respect to the norm \(\| \cdot \|\).

This terminology is justified by the following theorem.

**Theorem 2.3.** The normed field \((\hat{F}, \| \cdot \|)\) is complete and \(F\) is dense in \(\hat{F}\).

**Proof.** First we prove the second statement. Let \((a_n)\) be a Cauchy sequence in \(F\). Then, \((\hat{a}_n)\) is a sequence of (constant) Cauchy sequences and we have
\[
\lim_{n,m \to \infty} \|a_n - a_m\| = 0.
\]
Hence, \((\hat{a}_n)\) converges to the class represented by \((a_n)\). Hence, \(F\) is dense in \(\hat{F}\).

Now let \((A_n)\) be a Cauchy sequence of Cauchy sequences in \(F\), hence a representative of a Cauchy sequence in \(\hat{F}\). Since \(F\) is dense and \(A_n\) is a Cauchy sequence for every \(n\), we know that there exists a Cauchy sequence \((\hat{a}_n)\) for all \(n\) such that

\[
A_n - (\hat{a}_n) < \frac{1}{n}.
\]

It follows that

\[
(a_n) - (A_n) = ((a_n) - (\hat{a}_n)) - (A_n - (\hat{a}_n))
\]

is a null sequence in \(\hat{F}\). Therefore,

\[
\lim_{n \to \infty} \| (a_n) - A_n \| = 0
\]

The only thing left to check is that the field operations of \(\hat{F}\) come from \(F\) in a continuous way.

**Proposition 2.4.** Let \((a_n)\) and \((b_n)\) be Cauchy sequences in \(F \subset \hat{F}\). Then,

\[
\lim_{n \to \infty} (a_n + b_n) = (\lim_{n \to \infty} a_n) + (\lim_{n \to \infty} b_n)
\]

\[
\lim_{n \to \infty} (a_n \cdot b_n) = (\lim_{n \to \infty} a_n) \cdot (\lim_{n \to \infty} b_n)
\]

**Proof.** From the proof of the above theorem follows

\[
\lim_{n \to \infty} (a_n + b_n) = (a_n + b_n) = (a_n) + (b_n) = (\lim_{n \to \infty} a_n) + (\lim_{n \to \infty} b_n)
\]

where we denote the class represented by \((a_n)\) also by \((a_n)\). Analog for multiplication. \(\square\)

### 3 The field of \(p\)-adic numbers \(\mathbb{Q}_p\)

**Definition 3.1.** Let \(p \in \mathbb{N}\) prime. For \(0 \neq x \in \mathbb{Q}\) we define the \(p\)-adic order of \(x\)

\[
\text{ord}_p(x) = \begin{cases} 
\max\{n \in \mathbb{N} | p^n \text{ divides } x\}, & \text{if } x \in \mathbb{Z} \\
\text{ord}_p(a) - \text{ord}_p(b), & \text{if } x = a/b, b \neq 0, a, b \in \mathbb{Z}
\end{cases}
\]

**Remark 3.1.** For \(x = a/b, y = c/d \in \mathbb{Q}\) we have

\[
\text{ord}_p\left(\frac{ac}{bd}\right) = \text{ord}_p(ac) - \text{ord}_p(bd)
\]

\[
= \text{ord}_p(a) + \text{ord}_p(c) - \text{ord}_p(b) - \text{ord}_p(d)
\]

\[
= \text{ord}_p\left(\frac{a}{b}\right) + \text{ord}_p\left(\frac{c}{d}\right)
\]
and if \(x \neq -y\)

\[
\ord_p \left( \frac{a}{b} + \frac{c}{d} \right) = \ord_p \left( \frac{ad + cb}{bd} \right) = \ord_p(ad + cb) - \ord_p(bd)
\]  

(2)

\[
\geq \min\{\ord_p(ad), \ord_p(cb)\} - \ord_p(b) - \ord_p(d)
\]

\[
= \min\{\ord_p(a) + \ord_p(d), \ord_p(c) + \ord_p(b)\} - \ord_p(b) - \ord_p(d)
\]

\[
= \min\{\ord_p(a) - \ord_p(b), \ord_p(c) - \ord_p(d)\}
\]

\[
= \min\{\ord_p\left( \frac{a}{b}, \frac{c}{d} \right)\}
\]

**Definition 3.2.** On \(\mathbb{Q}\) we define the p-adic norm

\[
|x|_p = \begin{cases} 
p^{-\ord_p(x)}, & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases}
\]

**Proposition 3.1.** \(|.|_p\) is a non-Archimedean norm on \(\mathbb{Q}\).

*Proof.* \(|x|_p = 0 \iff x = 0\) follows from the definition of the \(|.|_p\) norm.

\(|xy|_p = |x|_p|y|_p\) follows from (1).

For the strong triangle inequality we have

\[
|x + y|_p = p^{-\ord_p(x+y)} \leq \max\{p^{-\ord_p(x)}, p^{-\ord_p(y)}\} = \max\{|x|_p, |y|_p\}
\]

from (2). \(\square\)

**Remark 3.2.** Unlike the euclidean norm on \(\mathbb{Q}\), given two numbers \(a, b \in \mathbb{Q}\) with \(|a|_p < |b|_p\), we can’t always find a third number \(c \in \mathbb{Q}\) so that \(|a|_p < |c|_p < |b|_p\). In particular, \(|.|_p\) only takes values in \(\{p^k | k \in \mathbb{Z}\} \cup \{0\}\).

**Definition 3.3.** Let \(p \in \mathbb{N}\) be prime. The field of p-adic numbers \(\mathbb{Q}_p\) is defined as the completion of \(\mathbb{Q}\) with respect to \(|.|_p\), and its elements are equivalence classes of Cauchy sequences.

For an element \(a \in \mathbb{Q}_p\) and a Cauchy sequence \((a_n)\) representing \(a\), we defined the norm of \(a\) as

\[
|a|_p = \lim_{n \to \infty} |a_n|_p
\]

**Remark 3.3.** By remark 3.2, the norm of \(a \in \mathbb{Q}_p\) only takes values in \(\{p^k | k \in \mathbb{Z}\} \cup \{0\}\), just like \(|.|_p\). Also, if \(|a|_p = p^k \neq 0\), then for any Cauchy sequence representing \(a\) there is an \(N\) so that \(|a_n|_p = p^k\) for \(n > N\).

How does an element of \(\mathbb{Q}_p\) look like? The following proposition will give a way to construct one of them.
Proposition 3.2. Let $d_m \neq 0$ and $0 \leq d_i < p$ integers. Then the partial sums of the series

$$a = d_mp^{-m} + d_{m+1}p^{-m+1} + \cdots + d_{-1}p + d_0 + d_1p + d_2p^2 + \ldots$$

form a Cauchy sequence and therefore $a$ is an element of $\mathbb{Q}_p$.

Proof. Let $\epsilon > 0$. Then we can find $N \in \mathbb{N}$ so that $p^{-N} < \epsilon$, and for $n, k > N$, WLOG $k > n$, we have

$$\left| \sum_{i=-m}^{k} d_ip^i - \sum_{i=-m}^{n} d_ip^i \right|_p = \left| \sum_{i=n+1}^{k} d_ip^i \right|_p \leq \max\{|d_{n+1}p^{n+1}|_p, \ldots, |d_kp^k|_p\} \leq p^{-N} < \epsilon$$

We might now wonder if every element of $\mathbb{Q}_p$ looks like a series (3).

The following propositions will help us prove that every $a \in \mathbb{Q}_p$ actually has a unique Cauchy sequence representing it that looks like (3).

Proposition 3.3. Let $x \in \mathbb{Q}$ with $|x|_p \leq 1$. Then for any $i$ there is a unique integer $\alpha \in \{0, 1, \ldots, p^i - 1\}$ so that $|x - \alpha|_p \leq p^{-i}$.

Proof. Let $x = a/b$ with $a$ and $b$ relatively prime. Since $|x|_p = p^{-\text{ord}_p(a)+\text{ord}_p(b)} \leq 1$ we get $\text{ord}_p(b) = 0$, that is, $b$ and $p^i$ are relatively prime for any $i$. We can then find integers $m$ and $n$ so that $np^i + mb = 1$. For $\alpha = am$ we get

$$|\alpha - x|_p = \left| am - \frac{a}{b} \right|_p = \left| \frac{a}{b} \right|_p |mb - 1|_p \leq |mb - 1|_p = |np^i|_p = |n|_pp^{-i} \leq p^{-i}$$

There is exactly a multiple $cp^i$ of $p^i$ so that $cp^i + \alpha \in \{0, 1, \ldots, p^i - 1\}$, and we have

$$|cp^i + \alpha - x|_p \leq \max\{|\alpha - x|_p, |cp^i|\} \leq \max\{p^{-i}, p^{-i}\} = p^{-i}$$

Theorem 3.4. Let $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$. Then there is exactly one Cauchy sequence $(a_n)$ representing $a$ so that for any $i$

1) $0 \leq a_i < p^i$
2) $a_i \equiv a_{i+1} \pmod{p^i}$

Proof. Let $(c_n)$ be a Cauchy sequence representing $a$. Since $|c_n|_p \rightarrow |a|_p \leq 1$, there is an $N$ so that $|c_n|_p \leq 1$ for any $n > N$. (If $|a|_p = 1$ this still holds because of remark 3.3)

By replacing the first $N$ elements we can find an equivalent Cauchy sequence so
that $|b_n|_p \leq 1$ for any $n$. Now, for every $j = 1, 2, \ldots$ let $N(j)$ be so that $N(j) \geq j$ and

$$|b_i - b_{i'}|_p \leq p^{-j} \quad \forall i, i' \geq N(j)$$

From the previous proposition we know that for any $j$ we can find integers $0 \leq a_j < p^j$ (condition i)) so that

$$|a_j - b_{N(j)}|_p \leq p^{-j}$$

These $a_j$ also satisfy condition ii):

$$|a_{j+1} - a_j|_p = |a_{j+1} - b_{N(j+1)} + b_{N(j+1)} - b_{N(j)} - b_{N(j)} - a_j|_p$$

$$\leq max\{|a_{j+1} - b_{N(j+1)}|_p, |b_{N(j+1)} - b_{N(j)}|_p, |b_{N(j)} - a_j|_p\}$$

$$\leq max\{p^{-j-1}, p^{-j}, p^{-j}\} = p^{-j}$$

This sequence is equivalent to $(b_n)$: for any $j$ take $i \geq N(j)$

$$|a_i - b_i|_p = |a_i - a_j + a_j - b_{N(j)} + b_{N(j)} - b_i|_p$$

$$\leq max\{|a_i - a_j|_p, |a_j - b_{N(j)}|_p, |b_{N(j)} - b_i|_p\}$$

$$\leq max\{p^{-j}, p^{-j}, p^{-j}\} = p^{-j}$$

so $|a_i - b_i| \rightarrow 0$.

Now, to show uniqueness, let $(d_n)$ be another Cauchy sequence satisfying conditions i) and ii) and let $(a_n) \neq (d_n)$, that is, for some $i_0$, $a_{i_0} \neq d_{i_0}$. Since $a_{i_0}$ and $d_{i_0}$ are between 0 and $p^{i_0}$, $a_{i_0} \neq d_{i_0} \pmod{p^{i_0}}$. From condition ii) we have that for $i > i_0$, $a_i = a_{i_0} \neq d_{i_0} = d_i \pmod{p^{i_0}}$, that is, $a_i \neq d_i \pmod{p^{i_0}}$ and therefore

$$|a_i - d_i|_p > p^{-i_0}$$

doesn’t converge to 0 and $(a_n)$ and $(d_n)$ aren’t equivalent.

**Remark 3.4.** For $a \in \mathbb{Q}_p$ with $|a|_p \leq 1$ we can write the Cauchy sequence $(a_n)$ representing $a$ from the previous proposition as

$$a_i = d_0 + d_1p + \cdots + d_{i-1}p^{i-1}$$

for $d_i \in \{0, 1, \ldots, p - 1\}$ and $a$ is represented by the convergent series

$$a = \sum_{i=0}^{\infty} d_i p^i$$

which we can think of as a number written in base $p$ which keeps extending to the left

$$a = \ldots d_n \ldots d_1 d_0$$
Remark 3.5. If \( a \in \mathbb{Q}_p \) with \( |a|_p = p^m > 1 \) then \( a' = ap^m \) satisfies \( |a'| = p^m p^{-m} = 1 \) and we can then write

\[
a = a'p^{-m} = p^{-m} \sum_{i=0}^{\infty} c_i p^i = \sum_{i=-m}^{\infty} d_i p^i
\]

with \( d_{-m} = c_0 \neq 0 \) and \( a \) becomes a fraction in base \( p \) with finitely many digits after the point and which extends infinitely to the left

\[
a = \ldots d_n \ldots d_1 d_0. d_{-1} d_{-2} \ldots d_{-m}
\]

Definition 3.4. This way of writing \( a \in \mathbb{Q}_p \) as a number written in base \( p \) which keeps extending to the left is called the \( p \)-adic expansion of \( a \). This will either look like

\[
a = \ldots d_n \ldots d_1 d_0
\]

for \( d_i \in \{0, 1, \ldots, p-1\} \), if \( |a|_p \leq 1 \), or like

\[
a = \ldots d_n \ldots d_1 d_0. d_{-1} d_{-2} \ldots d_{-m}
\]

for \( d_i \in \{0, 1, \ldots, p-1\} \) and \( d_{-m} \neq 0 \), if \( |a|_p > 1 \).