P-adic Functions - Part 1

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22.11.2011

1 Locally constant functions

Motivation: Another big difference between p-adic analysis and real analysis is the existence of nontrivial locally constant functions. We will look at some interesting properties of such functions.

Definition 1. *A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is called continuous at the point $a \in \mathbb{Z}_p$ if $\forall \epsilon > 0$ there $\exists \delta > 0$ such that $|x - a|_p < \delta$ implies $|f(x) - f(a)|_p < \epsilon, \forall x \in \mathbb{Z}_p$.*

A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous if it is continuous at all points $\forall a \in \mathbb{Z}_p$.

A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is uniformly continuous if $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - y|_p < \delta$ implies $|f(x) - f(y)|_p < \epsilon, \forall x, y \in \mathbb{Z}_p$.

Example 1. *Since the space $\mathbb{Z}_p$ is totally disconnected, the charasteristic function of any ball $U \in \mathbb{Z}_p$:*

$$\xi_U(x) : \begin{cases} 0, \text{ if } x \in U \\ 1, \text{ if } x \in \mathbb{Z}_p \backslash U \end{cases}$$

is continuous. This is clear since both the ball U and its complement $\mathbb{Z}_p \backslash U$ are open.

Definition 2. *A function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is called locally constant if $\forall x \in \mathbb{Z}_p$ there exists an open neightbourhood U_x of x (e.g. a ball of radius p^{-m} for some $m \in \mathbb{N}$ centered at x, $U_x = \{y \in \mathbb{Z}_p ||x - y|_p < p^{-m}\}$) such that f is constant on U_x.*

Example 2. *For $\alpha \in \mathbb{Z}_p$ we have its p-adic expension*

$$\alpha = \alpha_0 + \alpha p + ... + \alpha_n p^n + ... \text{ , } \alpha_n \in \mathbb{Z}, 0 < \alpha_n \leq (p - 1)$$

Define

$$f_n : \mathbb{Z}_p \to \mathbb{Q}_p$$

$$f_n(\alpha) = \alpha_n, \forall n$$

These functions are locally constant, because f_n remains unchanged if we replace α by any β with

$$|\beta - \alpha|_p < \frac{1}{p^n}$$

Furthermore we can extend these functions to the space $\mathbb{Q}_p$.

Theorem 1. *Locally constant functions are continuous.*

Proof. Result follows directly from Definition 2. □

Theorem 2. *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a locally constant function. Then $\mathbb{Z}_p$ can be written as the union*

$$\mathbb{Z}_p = \bigcup_{i=1}^{k} U_{x_i}$$

of finitely many disjoint balls such that the function f is constant on each of these balls. In particular, the set $\{f(x)|x \in \mathbb{Z}_p\}$ of all values assumed by f on $\mathbb{Z}_p$ has only finitely many distinct values.

Proof. Let U_x be defined like in Definition 2. These balls form a cover of $\mathbb{Z}_p$. By compactness of $\mathbb{Z}_p$, this cover contains a finite subcover $U_{x_1}, U_{x_2}, ..., U_{x_k}$. Furthermore, we know that two balls in a ultra-metric space are either disjoint or contained in one another. So if we delete all the balls which lie inside other balls, we obtain a cover of $\mathbb{Z}_p$ by disjoint balls. □

Theorem 3. *Any locally constant function in $\mathbb{Z}_p$ is uniformly continuous.*

Proof. Let

$$\mathbb{Z}_p = \bigcup_{i=1}^{k} U_{x_i}$$

be a partition on $\mathbb{Z}_p$ into disjoint balls as in Theorem 2. Let p^{-m_i} be the radius of U_{x_i}, i = 1,2,...,k, $m = max(m_i)$ and $\delta = p^{-m}$. We prove that for $\forall \epsilon > 0$ the chosen δ works:
Suppose that $|x - y|_p < \delta = p^{-m}$. Since $x \in U_{x_i}$ for some i and each point of the ball is its center, we may assume $x = x_i$. Then $|x_i - y|_p < p^{-m} \leq p^{-m_i}$ which means that $f(y) = f(x_i) = f(x)$. □

Definition 3. *Let $E \subset \mathbb{Z}_p$, not necessarily compact. A function $f : E \to \mathbb{Q}_p$ is called a step function on E if there exists a positive integer t such that*

$$f(x) = f(x_0) \text{ for all } x, x_0 \in E \text{ with } |x - x_0|_p \leq p^{-t}.$$

The smallest integer t for which this property holds is called the order of f.

Remark: It is clear from the definition that a step function is uniformly continuous and also locally constant on E. On $\mathbb{Z}_p$ (or on any compact set) it also holds, that any locally constant function is a step function.

Remark: In p-adic analisys, the step functions have a really surprising property. But first, for a positive integer t, we want to construct an explicit partition of $E \subset \mathbb{Z}_p$. Let $\mathbb{N}_t = \{0, 1, ..., p^t - 1\}$. For each $x \in \mathbb{Z}_p$ we write its canonical expansion :

$$x = x_0 + x_1 p + ... + x_{t-1} p^{t-1} + x_t p^t + ...$$

and we define

$$N_x = x_0 + x_1 p + ... + x_{t-1} p^{t-1}.$$

It follows that

$$N_x \in \mathbb{N}_t \text{ and } |x - N_x|_p \leq p^{-t}$$

Let

$$U(N, t) = \{x \in \mathbb{Z}_p | |x - N_x|_p \leq p^{-t} < p^{-t+1}\}.$$

For each $N \in \mathbb{N}_p$ we define $E(N) = E \cap U(N, t)$ We already seen that any $x \in \mathbb{Z}_p$ belongs to some $U(N, t)$ and since for any $N, M \in \mathbb{N}_t$ we have $|N - M|_p > p^{-t}$, it follows that the balls $U(N, t)$ are disjoint. Therefore:

$$E = \bigcup_{N=0}^{p^t - 1} E(N)$$

is a partition of E.

Theorem 4. *Any step function on $\mathbb{N}$ or $\mathbb{Z}_p$ is periodic.*

Proof. Let $E = \mathbb{N}$ or $E = \mathbb{Z}_p$ and $f : E \to \mathbb{Q}_p$ be a step function of order t. Consider the partition o E:

$$E = \bigcup_{N=0}^{p^t - 1} E(N)$$

If $x, y \in E(N)$ we have $|x-y|_p = |(x-N)+(N-y)|_p \leq p^{-t}$ by the strong triangle inequality and hence $f(x) = f(y)$. Notice that if $x \in E(N)$, then $x+p^t \in E(N)$. Therefore

$$f(x+p^t) = f(x) \text{ for } x \in E,$$

which means f is periodic. □

Remark: In real analysis continuous functions on closed intervals can be uniformly approximated by step functions. In p-adic analysis we have a similar result and additionally, p-adic step functions are continuous.

Theorem 5. *Let $E = \mathbb{N}$ or $E = \mathbb{Z}_p$. A function $f : E \to \mathbb{Q}_p$ is uniformly continuous on E if and only if for every positive integer s there exists another positive integer t=t(s) and a step function $S : E \to \mathbb{Q}_p$ of order at most t such that:*

$$|f(x) - S(x)|_p \leq p^{-s} \ , \forall x \in E$$

.

Proof. " $\Longrightarrow$ " Assume that f is uniformly continuous on E. Choose the two positive integers s and t=t(s) such that:

$$|f(x) - f(x_0)|_p \leq p^{-s} \text{ for } x, x_0 \in E \text{ with } |x - x_0|_p \leq p^{-t}$$

For x we have its p-adic expansion:

$$x = x_0 + x_1 p + ... + x_{t-1} p^{t-1} + x_t p^t + ...$$

and we define

$$N_x = x_0 + x_1 p + ... + x_{t-1} p^{t-1}.$$

Define the function $S : E \to \mathbb{Q}_p$ by

$$S(x) = f(N_x) \text{ if } x \in E.$$

S is a step function of order at most t. Then we get :

$$|f(x) - S(x)|_p = |f(x) - f(N_x)|_p \leq p^{-s},$$

which is what we want.
" $\Longleftarrow$ " Choose f and S like in our assumption. If x_0 satisfies:

$$|x - x_0|_p \leq p^{-t}.$$

Then we get

$$S(x) = S(x_0)$$
$$|f(x) - S(x)|_p \leq p^{-s}$$
$$|f(x_0) - S(x_0)|_p \leq p^{-s}$$

resulting:

$$|f(x) - f(x_0)|_p = |(f(x) - S(x)) - (f(x_0) - S(x_0))|_p \leq p^{-s}$$

from which follows that f is uniformly continuous.

□

2 Continuous and uniformly continuous functions

In the following let $E \subset \mathbb{Z}_p$ and let $x_0 \in E$ be an accumulation point of E.

Theorem 6. *Let $f : E \to \mathbb{Q}_p$, $g : E \to \mathbb{Q}_p$.*
(1) f is continuous at $x_0 \in E$ if and only if for every sequence $\{x_n\}$ satisfying $\lim_{x\to\infty} x_n = x_0$ we have

$$\lim_{x\to\infty} f(x_n) = f(x_0).$$

(2) If f and g are continuous at $x_0 \in E$ the so are $f + g$, $f - g$, fg, and if $g(x_0) \neq 0$ then so is f/g.

Proof. The proof is very similar to the real case. □

Example 3. *Let $f : \mathbb{N} \to \mathbb{Q}_p$ be defined as*

$$f(x) = \frac{1}{x - c}$$

with $c \in \mathbb{Z}_p$. If $c \notin \mathbb{N}$, then the denominator does not vanish and therefore, by the above theorem, f is continuous on $\mathbb{N}$. However f is not bounded on $\mathbb{N}$ - since c is a p-adic integer, we can find elements in $\mathbb{N}$ for which $|x-c|_p$ is arbitrarily small, hence $|f(x)|_p$ is arbitrarily large. If $c \in \mathbb{N}$ then f is not continuous at the point c.

Theorem 7. *Every continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is uniformly continuous and bounded on $\mathbb{Z}_p$.*

Proof. The theorem followings directly from compactness of $\mathbb{Z}_p$. □

Remark: Similar to real analysis we have the following extension result.

Theorem 8. *Let E be a subset of $\mathbb{Z}_p$ and let $\bar{E}$ be its closure. Let $f : E \to \mathbb{Q}_p$ be a uniformly continuous on E. Then there exists a unique function $F : \bar{E} \to \mathbb{Q}_p$ uniformly continuous and bounded on $\bar{E}$ such that*

$$F(x) = f(x), \text{ for } x \in E.$$

Proof. Let $y \in \bar{E}$. Then there exists a sequence $\{x_n\}$ in E such that

$$x_n \to y \text{ for } n \to \infty$$

Suppose $y \notin E$ (if $y \in E$ it's clear). Since f is uniformly continuous on E, for any positive integer s there exists another positive integer t=t(s) such that

$$(*)\ |f(x) - f(x_0)|_p \le p^{-s} \text{ for } x, x_0 \in E \text{ with } |x - x_0|_p \le p^{-t}$$

is satisfied. Because $x_n \to y$ there is an integer $N = N(t)$ such that

$$|x_n - y|_p \le p^{-t} \text{ for } \forall n \ge N.$$

Therefore fo $n, m \ge N$ we also have

$$|x_m - x_n|_p = |(x_m - y) - (x_n - y)|_p \le p^{-t},$$

and hence

$$|f(x_m) - f(x_n)|_p \le p^{-s}.$$

This means that $\{f(x_n)\}$ is a p-adic Cauchy sequence, and since $\mathbb{Q}_p$ is complete, it's limit $L = \lim_{n\to\infty} f(x_n) \in \mathbb{Q}_p$.
It is easy to see that the limit does not depend on the chosen sequence $x_n \to y$. Indeed, let $\{z_n\}$ be another sequnce such that $z_n \to y$. Then $\{x_n - z_n\}$ will be a null sequence and by the uniform continuity of f, $\{f(x_n) - f(z_n)\}$ is also a null sequence, but this implies that

$$L = \lim_{n\to\infty} f(z_n).$$

Thus the function $F : \bar{E} \to \mathbb{Q}_p$ given by

$$F(y) = \lim_{n\to\infty} f(x_n)$$

whenever $y \in \bar{E}$ and $y = \lim_{n\to\infty} x_n \in E$ and $x_n \in E$ is well defined.
Now we show that the function F is uniformly continuous on $\bar{E}$. Let y and y_0 be two points in $\bar{E}$ satisfying $|y - y_0|_p \le p^{-t}$. Choose x and x_0 in E such that:

$$|x - y|_p \le p^{-t}$$

$$|x_0 - y_0|_p \leq p^{-t}$$
$$|f(x) - f(y)|_p \leq p^{-s}$$
$$|f(x_0) - f(y_0)|_p \leq p^{-s}$$

It follows that
$$|x - x_0|_p = |(x - y) + (y - y_0) - (x_0 - y_0)|_p \leq p^{-t}$$
hence by (*) we have $|f(x) - f(x_0)|_p \leq p^{-t}$.
Therefore:

$$|F(y) - F(y_0)|_p = | - (f(x) - F(y)) + (f(x) - f(x_0)) + (f(x_0) - F(y_0))|_p \leq p^{-s}$$

proving the uniform continuity of F on $\bar{E}$.
Finally, F is bounded on $\bar{E}$. Indeed, otherwise there would exist an infinite sequence $\{y_n\} \subset \bar{E}$ such that
$$\lim |F(y_n)|_p = \infty.$$

Since E and hence $\bar{E}$ are subsets of a compact set $\mathbb{Z}_p$, there exists a subsequence $\{y_{r_n}\}$ such that the limit
$$y_0 = \lim_{n \to \infty} y_{r_n}$$

exists. Since all points are in $\bar{E}$ and since $\bar{E}$ is a closed set, we have $y_0 \in \bar{E}$. Now F is uniformly continuous on $\bar{E}$ and therefore continuous at y_0. But this implies

$$\lim_{n \to \infty} F(y_{r_n}) = F(y_0)$$

contrary to $\lim_{n \to \infty} |F(y_n)|_p = \infty$.
To prove the uniqueness of F, we assume that there is a second function G with the same proprieties. Then F-G is uniformly continuous on $\bar{E}$ and identically 0 on E. Since E is dense in $\bar{E}$, by continuity $F - G$ is also identically 0 on $\bar{E}$. □

Theorem 9. *Let $E \subset \mathbb{Z}_p$. If $\{f_n\}$ is a sequence of continuous functions on E. If $f_n \to f$ converges uniformly on E, then f is continuous on E.*

Proof. Similar to the real case. □

P-ADIC ANALYSIS COMPARED WITH REAL CONTINUITY AND DIFFERENTIABILITY

STEPHAN TORNIER

1. Points of Continuity

When studying maps from a topological space X to itself one may ask whether there is a map that is continuous at every point of a given subset $E \subseteq X$ and discontinuous at every point of its complement E^c. For instance, if $E = X$ the identity will do. Choosing $E = \emptyset$ amounts to finding a map that is discontinuous at every point of X. In case $X \in \{\mathbb{R}, \mathbb{Q}_p\}$, further distinguished subsets are $E = \mathbb{Q}$, which in both cases is countable dense, and $E = \mathbb{Q}^c$.

It turns out that for $E \in \{\emptyset, \mathbb{Q}^c\}$ such maps may be constructed whereas for $E = \mathbb{Q}$ it is impossible.

Proposition 1.1. Let $X \in \{\mathbb{R}, \mathbb{Q}_p\}$ and $E \in \{\emptyset, \mathbb{Q}^c\}$. There is a map from X to itself whose set of points of continuity is E.

Proof. Let $E = \emptyset$. The map $\chi_{\mathbb{Q}}$ will do for both $\mathbb{R}$ and $\mathbb{Q}_p$. For $\mathbb{Q}_p$ note that both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{Q}_p$ whence the preimage of an open set containing either $0 \in \mathbb{Q}_p$ or $1 \in \mathbb{Q}_p$ but not both cannot contain an open set.

Next, let $E = \mathbb{Q}^c$. For $X = \mathbb{R}$ consider the Thomae map $t : \mathbb{R} \to \mathbb{R}$ defined by

$$t : x \mapsto \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ is reduced} \\ 0 & x \in \mathbb{Q}^c \end{cases}.$$

Its set of points of continuity is exactly $\mathbb{Q}^c$. For $X = \mathbb{Q}_p$, consider

$$f : x \mapsto \begin{cases} \frac{1}{\nu(x)^2+1} & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$$

where $\nu(x)$ denotes the index in the p-adic expansion of x at which the latter's period has repeated once. Then f is discontinuous at every point in $x \in \mathbb{Q}$: we may choose a sequence $(x_n)_{n\in\mathbb{N}}$ of non-rational p-adic numbers converging to x due to the fact that $\mathbb{Q}^c$ is dense in $\mathbb{Q}_p$. Then $0 = \lim_{n\to\infty} f(x_n) \neq f(x) > 0$. However, f is continuous at every point of $x \in \mathbb{Q}^c$: if $(x_n)_n$ is a sequence of p-adic numbers converging to x then the image sequence $(f(x_{n_m}))_m$ of its subsequence $(x_{n_m})_m$ of rational numbers converges to zero as the index up to which the p-adic expansions of x and x_{n_m} coincide grows as m tends to infinity. Hence $\nu_{x_{n_m}} \to \infty$ as well. □

The impossibility of such a construction for $E = \mathbb{Q}$ may be viewed as a consequence of the Baire Category Theorem and is elaborated on subsequently. Recall [2] that complete metric spaces are Baire spaces. In particular, a complete metric space cannot be expressed as a countable union of nowhere dense sets.

Proposition 1.2 ($E = \mathbb{Q}$). Let $X \in \{\mathbb{R}, \mathbb{Q}_p\}$. There is no map from X to itself whose set of points of continuity is $\mathbb{Q}$.

Date: November 24, 2011.

The proof of proposition 1.2 is based on the following property of the set of points of continuity.

Definition 1.3. A subset of a topological space is of type G_δ if it is a countable intersection of open sets, and of type F_σ if it is countable union of closed sets.

As a mnenomic device use e.g. G–Gebiet, δ–Durchschnitt, F–fermé and σ–sum.

Proposition 1.4. Let X and Y be metric spaces. The set of points of continuity of a map $f : X \to Y$ is of type G_δ. Its set of points of discontinuity is of type F_σ.

Proof. It suffices to prove one assertion. For any subset A of X, define the oscillation of f on A by

$$\omega_f(A) := \sup\{d(f(x), f(y)) \mid x, y \in A\} \in \mathbb{R} \cup \{\infty\}$$

and the oscillation of f at a point $x \in X$ by

$$\omega_f(x) := \lim_{\varepsilon \to 0} \omega_f(B(x, \varepsilon))$$

if the limit exists. Then f is continuous at $x \in X$ if and only if $\omega_f(x) = 0$ whence its set of points of continuity equals

$$\{x \in X \mid \omega_f(x) = 0\} = \bigcap_{n \in \mathbb{N}} \{x \in X \mid \omega_f(x) < 1/n\}.$$

The assertion follows as the sets $\{x \in X \mid \omega_f(x) < 1/n\},\ n \in \mathbb{N}$ are open. □

Baire's Theorem for complete metric spaces and proposition 1.4 now imply the following result of which proposition 1.2 appears as an immediate corollary.

Proposition 1.5. Let X be a complete metric space and Y a metric space. Then there is no map $f : X \to Y$ whose set of points of continuity E is countable dense.

Proof. Assume such a map f exists. Then its set of points of discontinuity is of type F_σ, i.e.

$$X - E = \bigcup_{n \in \mathbb{N}} F_n$$

for closed sets $F_n, n \in \mathbb{N}$. Then each F_n is nowhere dense. Otherwise, $X - E$ would contain an open ball contradicting the assumption that E is dense in X. But E is the union of its points which are nowhere dense subsets of X as well. Hence X is a countable union of nowhere dense sets in contradiction to being a Baire space. □

2. Differentiability

In this section we elaborate on whether Rolle's Theorem, the Mean Value Theorem and the Inverse Function Theorem from elementary calculus carry over the p-adic setting, see Katok [1, Ch. 4.4].

Theorem 2.1 (Rolle's Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous. If f is differentiable on (a, b) and $f(a) = f(b)$ there is some $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Apart from the fact that $\mathbb{Q}_p$ cannot be made into an ordered fields — whence there is no notion of interval — there is no hope for a version of Rolle's theorem to hold due to the following example.

Proposition 2.2. There is a differentiable map $f : \mathbb{Z}_p \to \mathbb{Q}_p$ satisfying both $f(0) = f(1)$ and $f'(x) \neq 0\ \forall x \in \mathbb{Z}_p$.

Proof. Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be defined by $x \mapsto x^p - x$. Then $f(0) = 0 = f(1)$ but $f'(x) = px^{p-1} - 1$ whence $f'(x) \in p\,\mathbb{Z}_p - 1\ \forall x \in \mathbb{Z}_p$. Hence $f'(x) \neq 0\ \forall x \in \mathbb{Z}_p$. □

In real analysis, an important consequence of Rolle's Theorem is the Mean Value Theorem which implies a criterion for a map to be constant.

Corollary 2.3 (Mean Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be continuous. If f is differentiable on (a, b) there is some $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Corollary 2.4. Let $f : (a, b) \to \mathbb{R}$ be differentiable with $f' \equiv 0$. Then f is constant.

As above, corollaries 2.3 and 2.4 would require an ordering but fail in a much stronger sense in the p-adic setting.

Proposition 2.5. There is an injective, differentiable map $f : \mathbb{Z}_p \to \mathbb{Z}_p \subset \mathbb{Q}_p$ whose derivative vanishes identically.

Proof. Let $f : \mathbb{Z}_p \to \mathbb{Z}_p \subset \mathbb{Q}_p$ be defined by

$$f : \sum_{k=0}^{\infty} a_k p^k \mapsto \sum_{k=0}^{\infty} a_k p^{2k}$$

For $x, y \in \mathbb{Z}_p$ we obtain

$$|f(x) - f(y)|_p = p^{-\operatorname{ord}(f(x)-f(y))} = p^{-2\operatorname{ord}(x-y)} = |x - y|_p^2$$

which implies injectivity as well as vanishing derivative. □

The Inverse Function Theorem fails in a rather devastating sense as well.

Theorem 2.6 (Inverse Function Theorem). Let $U \subseteq \mathbb{R}$ be open and let $f : U \to \mathbb{R}$ be C^1. If $f'(a) \neq 0$ for some $a \in U$ then f is a local diffeomorphism at a.

Proposition 2.7. There is a continuously differentiable map $f' : \mathbb{Z}_p \to \mathbb{Q}_p$ with nowhere vanishing derivative which is injective in no neighbourhood of $0 \in \mathbb{Z}_p$.

Proof. For each $n \in \mathbb{N}$, let

$$B_n := B_{p^{-2n}}(p^n) = \{x \in \mathbb{Z}_p \mid |x - p^n|_p < p^{-2n}\} \subset \mathbb{Z}_p\,.$$

Since $x \in B_n$ implies $|x|_p = p^{-n}$ we have $B_n \cap B_m = \emptyset$ whenever $n \neq m$. Define

$$f : x \mapsto \begin{cases} x - p^{2n} & \text{if } x \in B_n \\ x & \text{if } x \in \mathbb{Z}_p - \bigcup_{n\in\mathbb{N}} B_n \end{cases}.$$

Then $f(p^n) = p^n - p^{2n}$ as $p^n \in B_n$. But $p^n - p^{2n}$ is contained in no B_m. Therefore $f(p^n - p^{2n}) = p^n - p^{2n}$ and f is injective in no neighbourhood of $0 \in \mathbb{Z}_p$.

As to differentiability, consider $g(x) := x - f(x)$. We aim to show that g' exists and vansishes everywhere in which case $f'(x) \equiv 1$. For a start, $g(x)$ is locally constant on $\mathbb{Z}_p - \{0\}$ and hence has the derivative $g'(x) = 0$ on $\mathbb{Z}_p - \{0\}$. It remains to check that $g'(0) = 0$. But

$$\left|\frac{f(x) - f(0)}{x}\right|_p = \begin{cases} \frac{|p^{2n}|_p}{|x|_p} = p^n & x \in B_n \\ 0 & x \in \mathbb{Z}_p - \bigcup_{n\in\mathbb{N}} B_n \end{cases}$$

whence the assertion. □

However, there is a stronger notion of differentiability than C^1 that helps restoring injectivity.

Definition 2.8 (Strict differentiability)*.* Let $E \subset \mathbb{Q}_p$ be non-empty without isolated points and let

$$\Phi f : (E \times E) - \Delta \to \mathbb{Q}_p,\ (x, y) \mapsto \frac{f(x) - f(y)}{x - y}$$

where $\Delta := \{(x, x) \mid x \in E\}$. Then f is called *strictly differentiable at* $a \in E$ if

$$\lim_{(x,y)\to(a,a)} (\Phi f)(x, y) = f'(a). \tag{1}$$

Remark 2.9*.* It follows from definition 2.8 that any strictly differentiable map is continuously differentiable. The converse does not hold in the p-adic setting as the proof of proposition 2.7 shows:

$$\lim_{n\to\infty} \frac{f(p^n) - f(p^n - p^{2n})}{p^{2n}} = 0 \neq 1 = f'(0).$$

For continuously differentibale maps from $\mathbb{R}$ to itself however, the mean value theorem implies the existence of the limit (1).

Proposition 2.10*.* Let $E \subset \mathbb{Q}_p$ be non-empty without isolated points and let $f : E \to \mathbb{Q}_p$ be strictly differentiable at $a \in E$. If $f'(a) \neq 0$ then f is injective on some neighbourhood of a.

Proof. We will construct a neighbourhood of $a \in \mathbb{Z}_p$ on which $f/f'(a)$ is an isometry, thus in particular injective. By definition of strict differentiability there exists some $\delta > 0$ such that for all $x \neq y \in \mathbb{Z}_p$ which satisfy $|x - a|_p \leq \delta$ and $|y - a|_p \leq \delta$ we have

$$\left| \frac{f(x) - f(y)}{x - y} - f'(a) \right|_p \leq |f'(a)|_p.$$

The isosceles triangle inequality then implies

$$\left| \frac{f(x) - f(y)}{x - y} \right|_p = |f'(a)|_p$$

whence $B_\delta(a)$ serves as the desired neighbourhood. □

References

1. S. Katok, *p-adic Analysis Compared with Real*, American Mathematical Society, 2007.
2. B. v. Querenburg, *Mengentheoretische Topologie*, Springer, 1979.