P-ADIC ANALYSIS COMPARED TO REAL. LECTURE 8

1. Antiderivatives. (Nina Otter)

In real analysis the theory of differentiation is closely related to the theory of integration through the fundamental theorem of calculus and the Radon Nykodim theorem.
Both theorems depend on the ordering of the real numbers. Since a non-archimedean field cannot be ordered, it is not surprising that none of these theorems hold in the non-archimedean case.
The fundamental theorem of calculus states that for every continuous function defined on a real interval $I = [a, b]$ and with values in $\mathbb{R}$ $\forall\, x \in [a, b]$ the function $F(x) := \int_a^x f(t)dt$ is continuous on $[a, b]$, differentiable on (a, b) and $\forall x \in (a, b)$ $F'(x) = f(x)$.
Thus the fundamental theorem of calculus guarantees that every continuous function $f : I \to \mathbb{R}$ has an *antiderivative*.

Remark: a function f is differentiable on a point a of its domain if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} =: f'(a)$ is finite. f' is called the *derivative* of f, f the *antiderivative* of f'.

This statement remains true if we substitute $\mathbb{R}$ by an ultrametric field K and I by a subset X of K with no isolated points.
This result is known as Dieudonné's theorem, by the name of the French mathematician Jean-Alexandre-Eugène Dieudonné (1906-1992), one of the initiators of the Bourbaki project and who gave important contributions in several fields of mathematics, among others functional analysis, abstract algebra and history of mathematics.

The derivative of a continuous function needs not be continuous. In a first year calculus course one learns that the derivative of the function

$$f(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

is continuous everywhere on $\mathbb{R}$ except for $x = 0$.
One would however never suspect how badly a derivative can be discontinuous.
In order to investigate this issue further we need a generalization of the concept of continuity.
A function $f : I \to \mathbb{R}$ is said to have the *intermediate value property* if whenever $c, d \in I$ and L is between $f(c)$ and $f(d)$, there exists a point z between c and d s.t. $f(z) = L$.
This property was believed by some mathematicians to be equivalent to the property of continuity, until in 1875 Jean-Gaston Darboux (1842-1917) showed that this was not true: he proved that every derivative has the intermediate value property and gave examples of

Date: November 29 2011.

very badly discontinuous derivatives.
Due to his fundamental contribution a function having the intermediate value property is called a *Darboux function.*

Lemma (Darboux)**.** *Suppose that* $f : [a, b] \to \mathbb{R}$ *is the derivative of a function. Then f has the intermediate value property.*

Proof. Let $c, d \in [a, b]$. Let L be between $f(c)$ and $f(d)$. W.l.o.g. $f(c) < L < f(d)$. Then g(x) = F(x)-Lx is differentiable on [c,d] and hence continuous and $g'(c) < 0 < g'(d)$.
Thus there exists $z \in (c, d)$ s.t. g takes on z its minimum value. Hence by Fermat's theorem $g'(z) = 0$ and $f(z) = L$ □

The converse does not hold.
Let $a \in [-1, 1]$ $\forall x \in [0, 1]$ define

$$f_a(x) = \begin{cases} \sin 1/x & \text{if } x \neq 0 \\ a & \text{otherwise} \end{cases} \tag{1}$$

then f_a is a Darboux function and for $a, a' \in [-1, 1]$, $f_a - f_{a'}$ is not a derivative if $a - a' \neq 0$ and thus f_a and $f_{a'}$ cannot be both derivative.
What we also learn from this examples is that the collection of all Darboux functions is not closed under addition.

Example (Example of a Darboux function discontinuous everywhere: Conway's base 13 function)**.** *Define a function* $f : [0, 1] \to \mathbb{R}$ *by* $f(0) = f(1) = 0$*. For* $x \in (0, 1)$ *expand* x *in base* 13*, that is, write* $x = \sum_{i=-s}^{-1} b_i 13^i$*, where* $s \in \mathbb{Z}_{\geqslant 2}$ *and* $b_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C\}$*.*
Define $f(x) = 0$ *unless there exist* $n, m \in \mathbb{Z}$ *s.t.* $n > m$ *and* $b_n = B$ *or* $b_n = C$ *and* $b_m = A$ *and* $\forall n'$ *s.t.* $m < n' < n$ *or* $n' < m$: $b_{n'} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$*. In this case for all* $l < n$ *let the map be the identity on* $b_l \in \{0, 1, 2, 3, 4, ..., 9\}$
and map:

$$b_n \to \begin{cases} - & \text{if } b_n = B \\ + & \text{if } b_n = C \end{cases}$$

and $x = \sum_{i=-s}^{-1} b_i 13^i \mapsto f(x) = \pm(\sum_{i=-s}^{-m-1} c_i 10^{i+m} + \sum_{i=-m+1}^{n-1} c_i 10^{i+m})$ *where* $c_i \in \{0, 1, 2, ..., 9\}$*.*

Example: x = 0,A198C7A7B01A765 f(x)=-1,765.

This function is a Darboux function:
let y *be a real number,* $\{c_n\}_{n\in\mathbb{Z}}$ *the coefficients of its expansion in base 10. Adding this coefficient to the tail end of the expansion in base 13 of any* $x \in (0, 1)$ *yields an* $x' \in (0, 1)$ *s.t.* $f(x') = y$*. More formally, if* $y = +\sum_{i=-r}^{t} c_i 10^i$ *(*$r, t \in \mathbb{Z}_{\geqslant 0}$*),* $x \in (0, 1)$*, then*

$$x' = \underbrace{\sum_{i=-s}^{-1} b_i 13^i}_{expansion\ of\ x} + C13^{-s-1} + \sum_{i=-s-2}^{-s-t-2} c_i 13^i + A13^{-s-t-3} + \sum_{i=-s-t-4}^{-s-t-4-r} c_i 13^i.$$

Example: let y = 8756,2 x=0,45AB77 then x'=0,45AB77C8756A2 $\in (0,1)$ *and* $f(x') = y$.
Thus we see that not only the function is surjective, but also that for every $y \in \mathbb{R}$ *there exists an x in every subinterval of* $[0,1]$ *s.t.* $f(x) = y$.
Hence the function assumes every real value on every subinterval of its domain and is discontinuous everywhere.

This function cannot be a derivative, in short we will see for what reason.

In a very precise sense there is nothing special about Darboux functions: every real function is the sum of two Darboux functions and also the pointwise limit of a sequence of Darboux functions. [1]

The second generalization we need in our investigation of derivatives of continuous functions is the concept of *Baire classes of functions*. A function is said to be of *Baire class 1* if it is the pointwise limit of a sequence of continuous functions. Inductively, a function is in *Baire class n*, for n a countable ordinal, if it is the pointwise limit of a sequence of functions of Baire class $n-1$.

Remark: during Lecture 7 sets of Baire category 1 where introduced. There is a connection between sets of Baire category 1 and functions of Baire class 1. Let (M,d) and (N,d') be metric spaces, (M,d) complete. If $f : M \to N$ is a function of Baire class 1 then $\{x \in M : f \text{ is continuous at } x\}^c$ is a set of Baire category 1. [2]

Clearly every continuous function is a Baire class 1 function. In both the real and non-archimedean case the derivative of a continuos function is a Baire class one function.

Lemma 1. *Let* $f : [a,b] \to \mathbb{R}$ *or* $f : X \to K$. *If f is differentiable then* f' *is a function of Baire class 1.*

Proof. We prove the real case first. Extend f to $[a, b+1)$ setting $f(x) = f(b)\ \forall x \in [b, b+1)$. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence s.t. $h_n \to 0$ as $n \to \infty$. $\forall n \in \mathbb{N}\ \forall x \in [a,b]$ define

$$f_n(x) := \frac{f(x+h_n) - f(x)}{h_n}.$$

Then $f_n : [a,b] \to \mathbb{R}$ is continuous and $\forall x \in [a,b]\ \lim_{n\to\infty} f_n(x) = f'(x)$.

Next we consider the non-archimedean case.
$\forall n \in \mathbb{N}\ \forall a \in X$ let $U_{a,n} := \{x \in X : | x - a | \leq 1/n\}$. Since X has no isolated points, $U_{a,n}$

[1] Bruckner, Differentiation of real functions (1994)
[2] Funktionalanalysis I und II, lecture notes by Michael Struwe (2007-2008).

can be written as a union of two disjoint nonempty and relatively clopen subsets U_1 and U_2. Thus $\forall n \in \mathbb{N}$ we can costruct a locally constant function

$$r_n : U_{a,n} \to K : x \mapsto \begin{cases} u_1, x \in U_1 \\ u_2, x \in U_2 \end{cases}$$

for fixed $u_1 \in U_1$ and $u_2 \in U_2$.
r_n is continuous and $\forall x \in X$ one has $| x - r_n(x) | \leq 1/n$ and $r_n(x) \neq x$.
Define $f_n : X \to K : x \mapsto \frac{f(r_n(x)-f(x)}{r_n(x)-x}$ then f_n is continuous and $\forall x \in X \lim_{n\to\infty} f_n(x) = f(x)$.

□

Summarizing, the derivative of a continuous real function is a Darboux function of Baire class 1. The converse does not hold: in example (1) the functions f_a are both Darboux and of Baire class 1 but they cannot be derivatives for every choice of a, since their sum can fail to be a derivative. Unfortunately, this is as far as we can get in the real case: a criterion for the existence of antiderivates does not seem to be known until now.[3]
At this point we can make an interesting observation: since a derivative is necessarily of Baire class 1, its set of continuity must be a dense set. Hence a derivative can be discontinous only to a certain degree – we see now why Conway's base 13 function cannot be a derivative. It is nevertheless possible for a derivative to be discontinuous on an uncountable set. Let $I = [a, b] \subset \mathbb{R}$ be an interval, $E \subset I$ a perfect nowhere dense subset s.t. $a, b \in E$ and let $[a, b] - E = \bigcup_{k\in\mathbb{N}}(a_k, b_k)$. Define $f : [a, b] \to \mathbb{R}$ by
$f(x) = \begin{cases} \frac{2(x-a_k)}{b_k-a_k} - 1, x \in [a_k, b_k] \\ 0, \quad \text{otherwise} \end{cases}$
then f is a derivative which is discontinuous at any point of E.

In the realm of non-archimedean fields things couldn't be rosier: there we have a complete characterization for the existence of antiderivatives. We will come back to this after the proof of Dieudonné's theorem in section 2.

2. Dieudonné's Theorem. (Berke Topacogullari)

As known from calculus, any continuous function has an antiderivative. This is actually one statement of the fundamental theorem of calculus: Given a continuous function f :

[3] W.H. Schikhof *Ultrametric calculus. An introduction to p-adic analysis*

$\mathbb{R} \to \mathbb{R}$, an antiderivative is given by

$$F(x) := \int_0^x f(t)\,dt.$$

Now we want to look at this question in the p-adic setting, and – to make our life a little bit easier – we will confine ourselves to functions defined on $\mathbb{Z}_p$. Explicitly, given a continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$, does it possess an antiderivative? As we will show, the answer is yes:

Theorem (Dieudonné)**.** *Any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ has a continuously differentiable antiderivative.*

However, since we have no analogue of the fundamental theorem of calculus, we have to construct antiderivatives in a completely different way. The idea is quite simple: We write our function f as a series $\sum f_n$, where the f_n are locally constant. If we now choose antiderivatives F_n for each f_n, our hope is that $\sum F_n$ is an antiderivative for f. Indeed, this will work, although we have to take care to choose the F_n in a good way.

Antiderivatives are of course not unique, as we can add any function $g \neq 0$ with $g' = 0$ to get another one. But in contrast to the situation in calculus, where all such g have to be constant, in p-adic analysis there are much more functions whose derivative is identically 0 – the locally constant functions for example. One might think at first that these are all, but this is not the case: Consider $g : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$g\left(\sum_{n=0}^{\infty} a_n p^n\right) := \sum_{n=0}^{\infty} a_n p^{2n}.$$

Let $x = \sum a_n p^n$ and $y = \sum b_n p^n$. If $x \neq y$, then there is a minimal index j such that $a_j \neq b_j$. Hence $|x-y|_p = p^{-j}$ and $|g(x)-g(y)|_p = p^{-2j}$, which we can write as

$$|g(x) - g(y)|_p = |x-y|_p^{\,2}.$$

Thus we see that g' vanishes everywhere on $\mathbb{Z}_p$. But it also follows that g is injective, so it is not even near to be locally constant.

Before we start with the proof of Dieudonné's theorem, it's maybe good to fix some notation: The (first) difference quotient $\Phi_1 f$ of f is defined as

$$\Phi_1 f : \mathbb{Z}_p \times \mathbb{Z}_p \setminus \{(x,x) : x \in \mathbb{Z}_p \times \mathbb{Z}_p\} \to \mathbb{Q}_p, \quad \Phi_1 f(s,t) := \frac{f(s) - f(t)}{s-t}.$$

Remember that f is said to be continuously differentiable[1] (or strictly differentiable or a C^1-function) if

$$\lim_{(x,y)\to(a,a)} \Phi_1 f(x,y) = f'(a) \quad \text{for all } a \in \mathbb{Z}_p,$$

and we will write $C^1(\mathbb{Z}_p \to \mathbb{Q}_p)$ for the set of all C^1-functions defined on $\mathbb{Z}_p$.

The first thing to do is to make sure, that a given continuous function can always be approximated uniformly by a sum of locally constant functions.

[1]Note that this term can have different meanings, depending on the author.

Lemma 2. *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be continuous. Then there are locally constant functions $f_n : \mathbb{Z}_p \to \mathbb{Q}_p$ such that*

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} f_n - f \right\|_\infty = 0.$$

Proof. Let $n \in \mathbb{N}$. From the strong triangle equality it follows readily that

$$x \sim y \quad :\Longleftrightarrow \quad |f(x) - f(y)|_p < n^{-1}$$

is an equivalence relation on $\mathbb{Z}_p$. After choosing a representative a_i for each equivalence class, we define

$$g_n(x) := f(a_i) \quad \text{if} \quad x \sim a_i.$$

These functions are locally constant (which is a direct consequence of the continuity of f) and we have

$$|g_n(x) - f(x)|_p < n^{-1} \quad \text{for all } x \in \mathbb{Z}_p.$$

We're interested in a series, so we set

$$f_1 := g_1 \quad \text{and} \quad f_n := g_n - g_{n-1} \text{ for } n \geq 2.$$

These functions are still locally constant, and since

$$\left\| \sum_{n=1}^{N} f_n - f \right\|_\infty = \|g_N - f\|_\infty = \sup_{x\in\mathbb{Z}_p} |g_N(x) - f(x)|_p \leq N^{-1},$$

we're done. □

The main point in the proof of Dieudonné's theorem is to know how to choose good antiderivatives, and the following lemma tells us, which conditions are sufficient.

Lemma 3. *Let $f_1, f_2, \ldots$ and f be continuous functions on $\mathbb{Z}_p$ such that $\sum f_n$ converges uniformly to f. Suppose each f_n has a continuously differentiable antiderivative F_n such that*

$$\max\{\|F_n\|_\infty, \|\Phi_1 F_n\|_\infty\} \leq \|f_n\|_\infty.$$

Then $\sum F_n$ converges uniformly to a C^1-function $F := \sum_{n=1}^{\infty} F_n$, which is an antiderivative of f.

Proof. First, we show that $\sum F_n$ converges uniformly: Since $\sum f_n$ converges uniformly, we know that $\lim \|f_n\|_\infty = 0$, and by assumption this leads to $\lim \|F_n\|_\infty = 0$. It is already clear now that $\sum F_n$ has to converges pointwise to F, but for $N \in \mathbb{N}$ we even have

$$\left\| \sum_{n=1}^{N} F_n - F \right\|_\infty = \left\| \sum_{n=N+1}^{\infty} F_n \right\|_\infty \leq \max_{n\geq N+1} \|F_n\|_\infty.$$

This term gets arbitrarily small and so $\sum F_n$ converges uniformly to F, which is consequently seen to be continuous.

Second, we show that F is a C^1-function and an antiderivative of f: So, let $a \in \mathbb{Z}_p$ and $\epsilon > 0$. Choose N such that $\|f_n\|_\infty < \epsilon$ for all $n \geq N$. Then we have for all $s, t \in \mathbb{Z}_p \times \mathbb{Z}_p, s \neq t$, and all $n \geq N$:

$$|\Phi_1 F_n(s,t) - f_n(a)|_p \leq \max\{\|\Phi_1 F_n\|_\infty, \|f_n\|_\infty\} = \|f_n\|_\infty < \epsilon.$$

Now, if $x, y \in \mathbb{Z}_p$ are sufficiently close to a, namely such that

$$|\Phi_1 F_n(x,y) - f_n(a)|_p < \epsilon \quad \text{for } n = 1, 2, \dots, N,$$

we eventually get

$$\begin{aligned} |\Phi_1 F(x,y) - f(a)|_p &= \left| \sum_{n=1}^{\infty} \Phi_1 F_n(x,y) - f_n(a) \right|_p \\ &\leq \sup_{n \in \mathbb{N}} |\Phi_1 F_n(x,y) - f_n(a)|_p \leq \epsilon. \end{aligned}$$

Since ϵ was arbitrary, F is continuously differentiable at a and $F'(a) = f(a)$. This is true for any a in $\mathbb{Z}_p$, so we're finished with the proof. □

As a last step we will show that every locally constant function on $\mathbb{Z}_p$ has an antiderivative satisfying the conditions mentioned in the previous lemma.

Lemma 4. *Let $g : \mathbb{Z}_p \to \mathbb{Q}_p$ be a locally constant function. Then g has a continuously differentiable antiderivative G satisfying*

$$\max\{\|G\|_\infty, \|\Phi_1 G\|_\infty\} \leq \|g\|_\infty.$$

Proof. For a suitable $n \in \mathbb{N}$, we can write g in the form

$$g(x) = \sum_{m=0}^{p^n-1} \lambda_m \xi_{m+p^n\mathbb{Z}_p},$$

where $\lambda_m \in \mathbb{Q}_p$ and $\xi_{m+p^n\mathbb{Z}_p}$ are the characteristic functions of $m + p^n\mathbb{Z}_p$, i.e.

$$\xi_{m+p^n\mathbb{Z}_p}(x) = \begin{cases} 1, & \text{if } x \in m + p^n\mathbb{Z}_p, \\ 0, & \text{otherwise.} \end{cases}$$

An antiderivative of g is given by

$$G(x) := \sum_{m=0}^{p^n-1} \lambda_m (x - m) \xi_{m+p^n\mathbb{Z}_p},$$

and it is trivial to check that G is a C^1-function.

Let $x \in m + p^n\mathbb{Z}_p$ for some m. Then $|x - m|_p \leq p^{-n}$ and hence

$$|G(x)|_p = |\lambda_m (x - m)|_p \leq |\lambda_m|_p.$$

It follows immediately that $\|G\|_\infty \leq \max |\lambda_m|_p = \|g\|_\infty$.

To show the remaining inequality

$$|\Phi_1 G(x,y)|_p \leq \|g\|_\infty \quad \text{for all } x, y \in \mathbb{Z}_p, x \neq y,$$

we consider two cases: First, let $x, y \in m + p^n\mathbb{Z}_p$ for some m. Then

$$|\Phi_1 G(x,y)|_p = \left|\frac{G(x)-G(y)}{x-y}\right|_p = \left|\frac{\lambda_m(x-y)}{x-y}\right|_p = |\lambda_m|_p \leq \|g\|_\infty .$$

Second, let $x \in m + p^n\mathbb{Z}_p$ and $y \in m' + p^n\mathbb{Z}_p$, where $m \neq m'$. Then $|x-m|_p \leq |x-y|_p$ and $|y-m'|_p \leq |x-y|_p$, and so

$$|\Phi_1 G(x,y)|_p = \left|\frac{G(x)-G(y)}{x-y}\right|_p = \left|\lambda_m \frac{x-m}{x-y} + \lambda_{m'}\frac{y-m'}{x-y}\right|_p \leq \|g\|_\infty .$$

Combining both cases leads to $\|\Phi_1 G\|_\infty \leq \|g\|_\infty$, which was the last thing we had to show. □

Putting everything together yields Dieudonné's theorem:

Proof of Dieudonné's theorem. By Lemma 2, we know that f can be written as $f = \sum f_n$, where the f_n are locally constant functions. Lemma 4 tells us that each f_n has an antiderivative $F_n \in C^1(\mathbb{Z}_p \to \mathbb{Q}_p)$, which satisfies $\max\{\|F_n\|_\infty, \|\Phi_1 F_n\|_\infty\} \leq \|f_n\|_\infty$. Now we just apply Lemma 3 and the theorem is proven. □

Of course – as in the real case – continuous functions are not the only functions having an antiderivative. However, it shouldn't be too surprising that in the p-adic case the situation can get much more weird. As a last point, we want to look at a concrete example.

Consider the following function on $\mathbb{Z}_p$:

$$f(x) := \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

We will show that f indeed has an antiderivative, given by

$$F\left(\sum_{n=0}^{\infty} a_n p^n\right) := \begin{cases} 0, & \text{if all } a_n \text{ vanish}, \\ a_{n_0}p^{n_0} + \ldots + a_{2n_0}p^{2n_0}, & \text{otherwise, where } a_{n_0} \text{ is the} \\ & \text{minimal coefficient} \neq 0. \end{cases}$$

Let $x = \sum_{n=n_0}^{\infty} a_n p^n$, where $a_{n_0} \neq 0$. Any $y \in \mathbb{Z}_p$ such that $|x-y|_p < p^{-2n_0}$ must have the same first $2n_0$ coefficients as x, and hence $F(y) = F(x)$. Thus F is locally constant on $\mathbb{Z}_p \setminus \{0\}$, and we have $F'(x) = 0 = f(x)$ there. It remains to look at the situation at 0: Let $y = \sum_{n=n_0}^{\infty} b_n p^n \neq 0$, where $n_0 \neq 0$. Then

$$|\Phi_1 F(y,0) - 1|_p = \frac{|F(y)-y|_p}{|y|_p} = p^{n_0}\left|\sum_{n=2n_0+1}^{\infty} b_n p^n\right|_p \leq p^{-n_0-1}.$$

This gets arbitrarily small if we choose y near enough at 0, and hence $F'(0) = 1$.

At this point we come back to our discussion of antiderivatives (Section 1) and will prove the following

Theorem. *Let K be a non-archimedean field, $X \subset K$ a subset with no isolated points. A function $f : X \to K$ has an antiderivative iff it is of Baire class* 1.

We will prove the weaker statement for $X = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$. Strengthening the assumptions, we can base our proof on the work done to prove Dieundonné's theorem. For the proof of the general case see Schikhof [1] or van Rooij [5].

Analogously to the case of continuous functions, the first thing we need to do in order to prove the necessarity of having an antiderivative is to approximate each function of Baire class 1 by locally constant functions.

Notation. $B_1(X) := \{f : X \to K : f \text{ is of Baire class } 1\}$
$T(X) := \{f : X \to K : f \text{ is a locally constant function}\}$

Lemma (Pointwise approximation of a Baire 1 function by locally constant functions)**.** *Let $f \in B_1(X)$. There exist $\{f_n\}_{n\in\mathbb{N}} \subset T(X)$ s.t. $\forall x \in X$ $\sum_{n\in\mathbb{N}} f_n(x) = f(x)$.*

Proof. Let $f \in B_1(X)$. f is the pointwise limit of a sequence of continuous functions $g_1, g_2, \ldots$. By lemma 2 (see section 2) every function $g \in C(X \to K)$ can be approximated by locally constant functions. Thus $\forall n \in \mathbb{N}$ there exists $f_n \in T(X)$ s.t. $\mid f_n - g_n \mid_\infty \leq 1/n$. Hence $\lim_{n\to\infty} f_n = f$. □

Now that this is proved, we can use the lemmas used in the proof of Dieudonné's theorem: let $f \in B_1(X)$ and $\sum_{n\in\mathbb{N}} f_n$ the pointwise approximation by locally constant functions. By lemma 3 we know that $\forall\ n$ $f_n \in T(x)$ has an antiderivative F_n satisfying the conditions of lemma 2 (it is an easy exercise to prove that if we substitute uniformly with pointwise in the statement of lemma 2 it still holds) and we get by lemma 2:
$\forall x \in X \qquad (\sum_{n\in\mathbb{N}} F_n(x))' = \sum_{n\in\mathbb{N}} f_n(x) = f(x)$.

References

[1] W.H. Schikhof *Ultrametric calculus. An introduction to p-adic analysis.* CUP, Cambridge, 1984

[2] Russell A. Gordon *The integrals of Lebesgue, Denjoy, Perron, and Henstock.* AMS, 1991

[3]Michael Struwe *Funktionalanalysis I und II, lecture notes.* Lectures held at ETH in the fall of 2007 and spring of 2008. http://www.math.ethz.ch/ struwe/Skripten/FA-I-II-26-8-08.pdf

[4] Andrew M. Bruckner *Differentiation of real functions.* AMS, 1994

[5] Arnoud C. M. van Rooij *Non-Archimedean functional analysis.* CRC, 1978