THE n-FOLD COMPLETE SEGAL SPACE Bord_n

1. Cobordism n-fold Segal spaces

We start with the definition of the cobordism category as in [1]:

Definition 1.1. Let V be a vector space. For every n-tuple $k_1, \ldots, k_n \geqslant 0$, we let $(\mathrm{PBord}_n^V)_{k_1,\ldots,k_n}$ be the collection of tuples $(M, (t_0^i \leqslant \ldots \leqslant t_{k_i}^i)_{i=1,\ldots n})$, where

(1) M is a closed n-dimensional submanifold of $V \times \mathbb{R}^n$,

(2) the composition $\pi : M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is a proper map,

(3) for every $S \subseteq \{1, \ldots, n\}$ and for every collection $\{j_i\}_{i \in S}$, where $0 \leqslant j_i \leqslant k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^n \to \mathbb{R}^S$ does not have $(t_{j_i})_{i\in S}$ as a critical value[1].

(4) for every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \ldots, t_n^i\}$, the map $p_{\{i+1,\ldots,n\}}$ is submersive at x.

This set is endowed with the topology coming from the topology on the set of embeddings into a given vector space using the quotient topology and the bijection

$$\coprod_M \mathrm{Emb}(M, V \times \mathbb{R}^n)/\mathrm{Diff}(M) \xrightarrow{\sim} \mathrm{Sub}(V \times \mathbb{R}^n).$$

Additionally, outside the box given by the t_j^i's everything is "sent off to infinity", i.e., we don't care what happens outside this box.

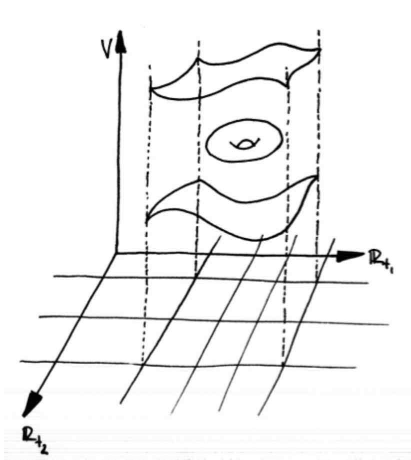
Remark 1.2. Let's unravel this definition for $n = 2$:

(2) $\Leftrightarrow$ "essential" part is compact

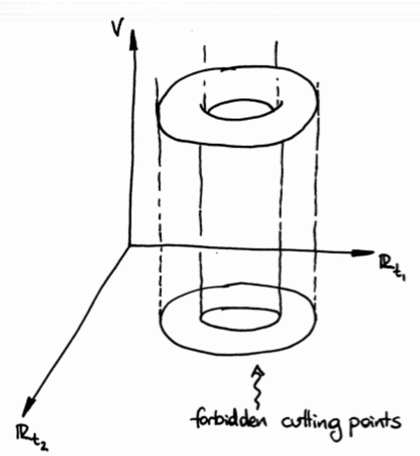
(3) $\Leftrightarrow$ cut at the "right" places

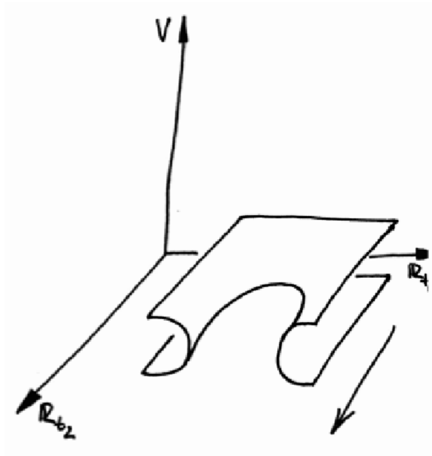
(4) $\Leftrightarrow$ gives "direction" of bordism

which we illustrate in the following figures:



[1]By critical value we mean that $\mathrm{d}p_S$ is surjective at the preimages of $(t_{j_i})_{i\in S}$.





The definition depends on the choice of the vector space V. However, in the bordism category we would like to consider all closed n-dimensional manifolds. Any such manifold can be embedded into some $V \times \mathbb{R}^n$, so we need to take the limit over all vector spaces V:

Definition 1.3.
$$\mathrm{PBord}_n = \varinjlim_V \mathrm{PBord}_n^V$$
Finally, we want to consider complete (n-fold) Segal spaces, so we need to complete our n-fold Segal space to obtain
$$\mathrm{Bord}_n .$$

Claim 1.4. $(\mathrm{PBord}_n)_{\bullet,\cdots,\bullet}$ *is an n-fold Segal space.*

Proof. It is clear that $(\mathrm{PBord}_n)_{\bullet,\cdots,\bullet}$ is an n-fold simplicial space. We need to check the following two conditions:

(i) for every i and for every $k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n \geqslant 0$,
$$(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},\bullet,k_{i+1},\ldots,k_n}$$
is a Segal space,

(ii) for every i and every $k_1, \ldots, k_{i-1}$,
$$(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,\bullet,\cdots,\bullet}$$
is essentially constant[2].

The first is the Segal condition which involves gluing. The proof is very similar to what we saw in Florian's talk. The second condition follows from condition (4) above:

Given an element $(M, (t_0^i \leqslant \ldots \leqslant t_{k_i}^i)_{i=1,\ldots n})$ in $(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,\bullet,\cdots,\bullet}$, we actually only care about a neighborhood of $p_{\{i\}}^{-1}(t_0^i)$, call this M'. Condition (4) implies that for every $x \in M$ such that $p_{\{i\}}(x) = t_0^i$, the map $p_{\{i+1,\ldots,n\}}$ is submersive at x, so it also is submersive in a neighborhood of x. Therefore

[2]Recall that an n-fold simplicial space $X_\bullet$ is essentially constant if there is a weak homotopy equivalence of n-fold simplicial spaces $Y \to X$, where Y is constant.

we can assume (by shrinking the chosen neighborhoods) that $p_{\{i+1,\ldots,n\}}$ is submersive in M', and thus, there is no critical point in M' and condition (3) gives no restriction on where to chose the t_j^l's for $l > i$. The space of the t_j^l's therefore is contractible, and $(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,\bullet,\cdots,\bullet}$ is weakly homotopy equivalent to the constant functor $(\mathrm{PBord}_n)_{k_1,\ldots,k_{i-1},0,\cdots,0}$. $\square$

2. Truncating Bord_n from above

The truncation of Bord_n (from above) is

$$\begin{aligned}\tau_k(\mathrm{Bord}_n) &= (\mathrm{Bord}_n)_{\underbrace{\bullet,\ldots,\bullet}_{k\text{ times}},\underbrace{0,\ldots,0}_{n-k\text{ times}}} \\ &\simeq \mathrm{Bord}_k\end{aligned}$$

as k-fold Segal spaces.

3. Truncating Bord_n from below resp. extending Bord_n up

Definition 3.1. Let $n \geqslant 0, l \leqslant n$. Let $(\mathrm{PBord}_n^l)_{k_1,\ldots,k_{n-l}}$ be the collection of tuples $(M, (t_0^i \leqslant \ldots \leqslant t_{k_i}^i)_{i=1,\ldots,n-l})$ satisfying

(1) M is a closed n-dimensional submanifold of $V \times \mathbb{R}^{n-l}$, and

(2)-(4) from definition 1.1.

Remark 3.2. This defines an $(n-l)$-fold Segal space, i.e. $(\infty, n-l)$-category, and is well-defined even for negative l. Note that l corresponds to the "level" we're extending down to, so $\mathrm{PBord}_n^0 = \mathrm{PBord}_n$, and $\mathrm{PBord}_n^{n-1} = \mathrm{PreCob}(n)$.

4. The homotopy bicategory $\mathrm{h}_2(\mathrm{Bord}_2)$

C. Schommer-Pries defined a two dimensional cobordism bicategory $2\,\mathrm{Cob}^{ext}$ in his thesis [2]. This bicategory turns out to be the homotopy bicategory of Bord_2. We first recall the definition of $2\,\mathrm{Cob}^{ext}$:

Definition 4.1. $2\,\mathrm{Cob}^{ext}$ is the bicategory such that

- objects are 0-dimensional manifolds
- 1-morphisms are 1-bordisms between objects, and
- 2-morphisms are isomorphism classes of 2-bordisms between 1-morphisms,

where

(1) A *1-bordism* between two 0-dimensional manifolds Y_0, Y_1 is a smooth compact 1-dimensional manifold with boundary W with a decomposition and isomorphism
$$\partial W = \partial_{in} W \sqcup \partial_{out} W \cong Y_0 \sqcup Y_1.$$

(2) A *2-bordism* between two 1-bordisms W_0, W_1 between objects Y_0, Y_1 is a compact 2-dimensional $<2>$-manifold S equipped with

- a decomposition and isomorphism

$$\partial_0 S = \partial_{0,in} S \sqcup \partial_{0,out} S \xrightarrow{\sim} W_0 \sqcup W_1,$$

- a decomposition and isomorphism

$$\partial_1 S = \partial_{1,in} S \sqcup \partial_{1,out} S \xrightarrow{\sim} Y_0 \times [0,1] \sqcup Y_1 \times [0,1].$$

Recall that a $<2>$-manifold is a manifold with faces X with a pair of faces $(\partial_0 X, \partial_1 X)$ such that

$$\partial_0 X \cup \partial_1 X = \partial X, \qquad \partial_0 X \cap \partial_1 X \text{ is a face.}$$

(3) Two 2-bordisms S, S' are *isomorphic* if there is a diffeomorphism $h : S \to S'$ compatible with the boundary data.

Vertical and horizontal compositions are defined by choosing collars and gluing.

Proposition 4.2. *There is an equivalence of bicategories between* $\mathrm{h}_2(\mathrm{Bord}_2)$ *and* $2\,\mathrm{Cob}^{ext}$.

Proof. By Whitehead's theorem for bicategories ([2]) it is enough to find a functor F which is

(1) essentially surjective on objects, i.e. F induces an isomorphism $\pi_0(\mathrm{h}_2(\mathrm{Bord}_2)) \cong \pi_0(2\,\mathrm{Cob}^{ext})$,
(2) essentially full on 1-morphisms, i.e. $F_{x,y} : \mathrm{h}_2(\mathrm{Bord}_2)(x,y) \to 2\,\mathrm{Cob}^{ext}(Fx, Fy)$ is essentially surjective, and
(3) fully-faithful on 2-morphisms, i.e. $F_{x,y} : \mathrm{h}_2(\mathrm{Bord}_2)(x,y) \to 2\,\mathrm{Cob}^{ext}(Fx, Fy)$ if fully-faithful.

Let

$$F : \mathrm{h}_2(\mathrm{Bord}_2) \longrightarrow 2\,\mathrm{Cob}^{ext}$$

be the functor defined as follows:

On objects,

$$(M \hookrightarrow V \times \mathbb{R}^2, t_0^1, t_0^2) \in (\mathrm{Bord}_2)_{0,0} \quad \stackrel{F}{\longmapsto} \quad \pi^{-1}\left((t_0^1, t_0^2)\right),$$

where the image is thought of as an abstract manifold. This is well-defined, because as π is proper and (t_0^1, t_0^2) is not a critical point of π, $\pi^{-1}\left((t_0^1, t_0^2)\right)$ is a finite disjoint union of points.

On 1-morphisms,

$$(M \hookrightarrow V \times \mathbb{R}^2, t_0^1 \leqslant t_1^1, t_0^2) \in (\mathrm{Bord}_2)_{1,0} \stackrel{F}{\longmapsto} \begin{cases} \pi^{-1}\left([t_0^1, t_1^1] \times \{t_0^2\}\right), \text{if } t_0^1 \neq t_1^1, \\ \pi^{-1}\left((t_0^1, t_0^2)\right) \times [0,1], \text{if } t_0^1 = t_1^1, \end{cases}$$

where again the image is thought of as an abstract manifold. Moreover, the decomposition of the boundary of the image is given by

$$\pi^{-1}\left((t_0^1, t_0^2)\right) \sqcup \pi^{-1}\left((t_1^1, t_0^2)\right)$$

respectively

$$\pi^{-1}\left((t_0^1, t_0^2)\right) \times \{0\} \sqcup \pi^{-1}\left((t_0^1, t_0^2)\right) \times \{1\}.$$

Well-definedness follows similarly to above.

A 2-morphisms $(M \hookrightarrow V \times \mathbb{R}^2, t_0^1 \leqslant t_1^1, t_0^2 \leqslant t_1^2) \in (\mathrm{Bord}_2)_{1,1}$ is sent to

- if $t_0^1 \neq t_1^1, t_0^2 \neq t_1^2$,
$$\pi^{-1}\left([t_0^1, t_1^1] \times [t_0^2, t_1^2]\right)$$
with the decomposition of the boundary coming from the inverse images under π of the sides of the rectangle $[t_0^1, t_1^1] \times [t_0^2, t_1^2]$. The source and target of our 2-bordism correspond to the top and bottom boundaries $W_0 \sqcup W_1$. Note that, because of condition (4), at every point in the fiber of the projection $p_{\{1\}} : M \to \mathbb{R}$ over $t_j^1 \in \{t_0^1, t_1^1\}$, the projection $p_{\{2\}}$ to the other axis is submersive. Therefore, the other components of the boundary of the 2-bordism are diffeomorphic to $Y_0 \times [0,1] \sqcup Y_1 \times [0,1]$, where $\partial W_i = Y_0 \sqcup Y_1, i = 0, 1$.
- if $t_0^1 = t_1^1, t_0^2 \neq t_1^2$,
$$[0,1] \times \pi^{-1}\left(\{t_0^1\} \times [t_0^2, t_1^2]\right)$$
with the decomposition of the boundary into source and target being
$$\{0\} \times \pi^{-1}\left(\{t_0^1\} \times [t_0^2, t_1^2]\right) \sqcup \{1\} \times \pi^{-1}\left(\{t_0^1\} \times [t_0^2, t_1^2]\right)$$
and additional trivial components
$$[0,1] \times \pi^{-1}\left((t_0^1, t_0^2)\right) \sqcup [0,1] \times \pi^{-1}\left((t_0^1, t_1^2)\right).$$
- if $t_0^1 \neq t_1^1, t_0^2 = t_1^2$, we define if analogous to the previous case,
- if $t_0^1 = t_1^1, t_0^2 = t_1^2$,
$$\pi^{-1}\left((t_0^1, t_0^2)\right) \times [0,1]^2$$
with the decomposition of the boundary coming from the sides of the rectangle $[0,1]^2$.

Note that any path in $(\mathrm{Bord}_s)_{1,1}$ induces an isomorphism of the corresponding 2-bordisms. So the map is well-defined on 2-morphisms.

Now we can check (1)-(3):

For (1), any point is the image of the plane $(\mathbb{R}^2 \hookrightarrow \mathbb{R}^2, 0, 0)$. For k points, we can take k disjoint parallel planes in $\mathbb{R} \times \mathbb{R}^2$ e.g. intersecting the $V = \mathbb{R}$ at the points $0, \dots, k-1$ and again choosing $t_0^1 = t_0^2 = 0$.

For (2), we use the classification of 1-dimensional manifold with boundary which we saw a couple of weeks ago when we discussed 1 Cob. Any connected component can be cut into pieces looking like straight lines and left and right half circles. Clearly we can first embed these into some $V \times \mathbb{R}_{t^1}$ such that its boundary is projected to some $t_0^1 < t_1^1$ in $\mathbb{R}_{t^1}$, and then we embed this into $V \times \mathbb{R}_{t^1} \times \{0\} \hookrightarrow V \times \mathbb{R}^2$. We can obviously "compose" these embeddings.

For (3), for showing that it is full on 2-morphisms, we use the classification theorem of Schommer-Pries [2]. He gives a set of generating 2-morphisms which obviously all are the image of an element in $(\mathrm{Bord}_2)_{1,1}$. Moreover, this construction behaves well with gluing. For faithfullness, we use the fact that

$$\mathrm{Emb}(M, V \times \mathbb{R}^n)$$

is contractible for V large enough, so $\mathrm{Emb}(M,V)/\mathrm{Diff}(M)$ is path connected. Moreover, since

$$\coprod_M \mathrm{Emb}(M, V \times \mathbb{R}^n)/\mathrm{Diff}(M) \longrightarrow \mathrm{Sub}(V \times \mathbb{R}^n)$$

is a bijection, where the coproduct is taken over all diffeomorphism classes of n dimensional manifolds, isomorphisms of 2-bordisms arise exactly as paths in $(\mathrm{Bord}_2)_{1,1}$. □

References

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[2] C. Schommer-Pries, The classification of two-dimensional extended topological field theories, Thesis (Ph.D.)–University of California, Berkeley, ProQuest LLC, Ann Arbor, MI, 2009.