Homotopical and Higher Algebra: detailed plan

Week 1

Symmetric monoidal categories

Definition and examples of monoidal categories (Sets, Vect, k-mod, Alg, …). Draw coherence diagrams and state Mac Lane coherence theorem. Same for symmetric monoidal categories (ex: chain complexes). Symmetric monoidal functors (ex: homology).

References: [9, Ch. XI] and [7, §3.2].

The symmetric monoidal category nCob, and nTFTs

General construction (oriented and unoriented versions). Detailed description of 1Cob and 1TFTs: classification theorem for 1-dim. manifolds with boundary (both versions).

References: [7, §1.1, 1.2 & 1.3] and [8, §1.1].

Week 2

Duality in monoidal categories

Left and right duals in a monoidal category. Equivalence of the two notions in the symmetric monoidal case. Examples of dualizable objects in Sets, Vect, k-mod and Alg. Property preserved by monoidal functors. Any object is dualizable in nCob.

Reference: [8] p.38

Presentation of 1Cob by generators and relations

Generators and relations for (symmetric) monoidal categories. Give a presentation by generators and relations for 1Cob. Conclude that 1TFTs with values in ℂ are dualizable objects in ℂ (in other words, 1Cob is the free symmetric monoidal category generated by a single dualizable object).

Reference: [7] §1.4]

Week 3

Explicit description of 2Cob

Classification theorem for oriented 2-dim manifolds with boundary. Pant decomposition and generators of 2Cob. Relations for 2Cob.

References: [7] §1.4] and [1].
Frobenius algebras and 2TFTs

Definition and examples of Frobenius algebras. Frobenius algebras are oriented 2TFTs.

References: [7, Ch. 2 and §3.3] and [11].

Week 4
Extending down TFTs

Classifying 3-manifolds. Problem in finding generators (i.e. problem of computing 3TFTs). Solution: manifolds with boundaries as generators. Extending down TFTs. Heuristic definition of 2-extended and fully extended TFTs. Need for higher categories.

References: [8, §1.2]

Bicategories

Definition, examples and first properties of bicategories (Cat, sets with correspondences, algebras with bimodules, 2Cob\text{ext}, the classifying 2-category of a monoidal category).

References: [3] and [10, Appendix B].

Week 5
Adjoint s in bicategories

Adjoint s in a bicategory (ex: usual adjoints in Cat, and dualizable objects in a monoidal category). Every 1-morphism in 2Cob\text{ext} has left and right adjoints.


Symmetric monoidal bicategories

Disjoint union in 2Cob\text{ext} is an additional structure. Extract from it the definition of a symmetric monoidal bicategory. Main examples of symmetric monoidal bicategories (Cat, sets with correspondences, algebras with bimodules, 2Cob\text{ext}, ...).

References: [10] Ch. 2.

Week 6
2-dualizable objects

Define 2-dualizable objects as objects having duals for which (co)evaluation have adjoints. Example: 2-dualizable objects in Alg\text{2} and relation with separable Frobenius algebras.


2-extended 2TFTs

Description of 2Cob\text{ext} by generators and relations. Relation with the free symmetric monoidal bicategory generated by a single 2-dualizable object.

References: [10] Ch. 3] and [8] pp.92-94].
Week 7

Extending up: ∞-categories

Summary of what we’ve done so far.
Explanation for why we need higher invertible arrows.
Overview of models for higher categories.

Week 8

Model categories

Definition and examples. Yoga of derived functors. Consider the main example of chain complexes.
Reference: [6].

Models for ∞-groupoids

Reference: [6].

Week 9

Models for (∞,1)-categories

Topological and simplicial categories. Quasi-categories and weak Kan complexes. Complete Segal spaces and Segal categories.
Reference: [4]

Quillen equivalences

Sketch of proof that all models are equivalent.
References: [4 §7] and [5].

Week 10

The (∞,1)-category 1Cob∞

Description of 1Cob∞ as a complete Segal space.

The homotopy category of 1Cob∞ is 1Cob

Topological and differentiable structures. Contractibility of the space of choices.

Week 11

Symmetric monoidal structure on 1Cob∞

Discuss symmetric monoidal structures on (∞,1)-categories. Examples: Chain complexes and dg-Alg.
1 TFT\(_\infty\)'s
Symmetric monoidal structure on 1Cob\(_\infty\). Description of 1 TFT\(_\infty\)'s with values in chain complexes and dg-Alg.
Reference: [8] beginning of §4.2

Week 12
Detailed study of 2 TFT\(_\infty\)'s. In particular, the homotopy category of 2Cob\(_\infty\) is 2Cob\(_\text{ext}\).

Week 13
Lurie's main theorem (fully dualizable objects).

References