The $(\infty,1)$-category $Cob_t(1)$

We use a new approach to study TFT's. [redacted] To do so we introduce a $(\infty,1)$-category which should encode the following data:

- closed $(n-1)$-manifolds
- n-bordisms between closed $(n-1)$-mfolds
- "compatible" diffeomorphisms between such n-bordisms ("compatible" means that they have to commute with the identifications on the boundary of our bordism)
- "compatible" isotopies between $\overset{\text{such}}{\frown}$ diffeomorphisms
- "compatible" isotopies between such isotopies
- etc.

This is a refinement of the data we have in the category $Cob(n)$

We will define our $(\infty,1)$-category as a complete Segal space. This requires some preliminary work.

The homotopy fiber product

Def: Let X,Y,Z be top. spaces, $f: X \to Z$, $g: Y \to Z$ continuous maps.

We define the homotopy fiber product as a subspace of $X \times_Z Z^{[0,1]} \times_Z Y$

with the maps $X \xrightarrow{f} Z \longleftarrow Z^{[0,1]} \longrightarrow Z \xleftarrow{g} Y$.
$\qquad\qquad\qquad\quad p(0) \longleftarrow\!\shortmid \; p \;\shortmid\!\longrightarrow p(1)$

$$X \times^R_Z Y := \{ (x,p,y) \in X \times_Z Z^{[0,1]} \times_Z Y \mid p \text{ is continuous} \}$$

Remark 1: There is a natural inclusion map $X \times_Z Y \longrightarrow X \times^R_Z Y$

$(x,y) \longmapsto (x,p,y)$ with $p(t) = f(x)$ $\forall t \in [0,1]$

Remark 2: If we have the following commuting diagramm

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ X' \arrow[r, "f'"] & Z' & Y' \arrow[l] \end{tikzcd}$$

[redacted] and the vertical maps are weak equivalences, then the induced map

$X \times^R_Z Y \longrightarrow X' \times^R_{Z'} Y'$ is a weak equivalence.

The semisimplicial category

Let Δ denote the simplicial category. We have a subcategory $\Delta_0 \subset \Delta$ given with $Ob(\Delta_0)$: nonnegative integers, denoted by $[n]$, $n \in \mathbb{N}_0$

$Mor_{\Delta_0}([k],[\ell])$: $f: \{0,\dots,k\} \to \{0,\dots,\ell\}$ strictly increasing. (composition = composition of maps)

We can now define the following:

A semisimplicial object of a category $\mathcal{C}$ is a functor from Δ_0^{op} to $\mathcal{C}$

If $\mathcal{C} = \mathcal{T}op$, then we call such a functor a semisimplicial space.

The weak Segal condition and SemiSegal spaces

Let $X_\bullet$ be a semisimplicial space. Let $m,n \geq 0$. We can now define the following morphisms in Δ_0:

$$p_{0,\dots,m}: \{0,\dots,m\} \longrightarrow \{0,\dots,m+n\}, \quad i \longmapsto i \qquad\qquad p_m: \{0\} \to \{0,\dots,m\}, \quad 0 \longmapsto m$$

$$p_{m,\dots,m+n}: \{0,\dots,n\} \longrightarrow \{0,\dots,m+n\}, \quad i \longmapsto i+m \qquad\qquad p_0: \{0\} \to \{0,\dots,n\}, \quad 0 \longmapsto 0$$

We have the commuting diagram

$$\begin{tikzcd} {[m+n]} & {[m]} \arrow[l, "p_{0,\dots,m}"'] \\ {[n]} \arrow[u, "p_{m,\dots,m+n}"] & {[0]} \arrow[l, "p_0"] \arrow[u, "p_m"'] \end{tikzcd}$$

Applying our functor – which is our semisimplicial space – we get another commuting diagram

$$\begin{tikzcd} X_{m+n} \arrow[r, "p_{0,\dots,m}^*"] \arrow[d, "p_{m,\dots,m+n}^*"'] & X_m \arrow[d, "p_m^*"] \\ X_n \arrow[r, "p_0^*"'] & X_0 \end{tikzcd}$$

This diagram induces a map to the fiber product which can be composed with the inclusion map into the homotopy fiber product.

$$X_{m+n} \longrightarrow X_m \times_{X_0} X_n \longrightarrow X_m \times^R_{X_0} X_n$$

Def: A semisimplicial space or a simplicial space $X_\bullet$ satisfies the weak Segal condition

$:\Leftrightarrow$ $\forall m,n \geq 0$ $\quad X_{m+n} \to X_m \times^R_{X_0} X_n$ is a weak equivalence.

Remark: The diagram above can be "iterated" by splitting up m and n into sums of smaller integers. If all these numbers are chosen positive we get a commutative diagram

$$\begin{tikzcd} X_{m+n} \arrow[rr] \arrow[dr] \arrow[dd] & & X_1 \arrow[dd] \\ & X_1 \arrow[d] & \\ X_1 \arrow[r] & {} & {} \end{tikzcd}$$

and a map $X_{m+n} \to \underbrace{X_1 \times_{X_0} \dots \times_{X_0} X_1}^{m+n}$

It is now easy to see that the weak Segal condition is satisfied if and only if for all $n>0$ the map $X_n \to \underbrace{X_1 \times_{X_0} \dots \times_{X_0} X_1}_{n}$ is a weak equivalence.

Construction of SemiCob(n).

With these tools we can now construct a semiSegal space. Let V be a real finite-dimensional vectorspace. Define the set $\mathrm{SemiCob}(n)_k^V := \{(M, t_0 < \dots < t_k) \mid t_i \in \mathbb{R},\ M \subseteq V \times [t_0, t_k]$ a compact, embedded n-dimensional submanifold with $\partial M = M \cap V \times \{t_0, t_k\}$ and $\forall i \in \{0, \dots, k\}: M \pitchfork V \times \{t_i\}\}$

for $k>0$ and $\mathrm{SemiCob}(n)_0^V := \{(M, t_0) \mid t_0 \in \mathbb{R},\ M \subseteq V$ a compact, closed submfld of V with no boundary$\}$

We define topologies on $\mathrm{SemiCob}(n)_k^V$ as follows: First note that as a set $\mathrm{SemiCob}(n)_k^V$ is in bijection with a subset of $\mathrm{Sub}(V) \times \mathbb{R}^{k+1}$ respectively $\mathrm{Sub}_0(V) \times \mathbb{R}$ for $k=0$, where

$$\mathrm{Sub}(V) := \{(M, f) \mid M \text{ a compact } n\text{-dim mfld, } f: M \to V \times [0,1] \text{ an embedding such that } f(\partial M) = f(M) \cap V \times \{0,1\}\}$$

$$\mathrm{Sub}_0(V) := \{M \mid M \subseteq V \text{ a } (n-1)\text{-dim. closed submfld}\}$$

If we give these spaces, we can give $\mathrm{SemiCob}(n)_k^V$ the unique topology such that the bijection – given by orientation-preserving affine transformation from $[t_0, t_k]$ to $[0,1]$ – to a subset of $\mathrm{Sub}(V) \times \mathbb{R}^{k+1}$ resp. $\mathrm{Sub}_0(V) \times \mathbb{R}$ is a homeomorphism w.r.t. the subspace topology.

The topology on $\mathrm{Sub}(V)$ is induced by the natural bijection $\coprod_{\substack{[M] \text{ diffeom. classes of closed} \\ n\text{-dim. mflds}}} \mathrm{Emb}(M, V \times [0,1]) / \mathrm{Diff}(M) \xrightarrow{\sim} \mathrm{Sub}(V)$

where $\mathrm{Emb}(M, V \times [0,1])$ carries the Whitney topology (see [1]) which makes the left-hand-side a topological space.

Analogously one can define a topology on $\mathrm{Sub}_0(V)$. This makes $\mathrm{SemiCob}(n)_k^V$ a top. space.

After defining the images of $[k]$ under the functor $\mathrm{SemiCob}(n)_\cdot^V$, we need to define the images of the morphisms. Let $f: \{0, \dots, k\} \to \{0, \dots, l\}$ be strictly increasing.

We define $f_V^*((M, t_0 < \dots < t_l)) = (M \cap V \times [t_{f(0)}, t_{f(k)}],\ t_{f(0)} < \dots < t_{f(k)})$

f^* is continuous w.r.t. the topologies defined earlier and completes the definition of our functor.

This functor depends on V. To get rid of V, we choose any infinite-dimensional real vectorspace $\mathbb{R}^\infty$.

Then we define $\mathrm{SemiCob}(n)_k := \varinjlim_{V \subseteq \mathbb{R}^\infty,\ V \text{ finite dimensional}} \mathrm{SemiCob}(n)_k^V$ and $f^* := \varinjlim_{V \subseteq \mathbb{R}^\infty,\ V \text{ finite dimensional}} f_V^*$

For this we took the partial order of inclusion on all linear subspace of V.

Claim: $\mathrm{SemiCob}(n)_\bullet$ is a semiSegal space.

Remark: For different choices of $\mathbb{R}^\infty$ we get homotopy equivalent space. If $\mathbb{R}^\infty$ has countable dimension,

then the directed limit is even unique up to homeomorphism.

The claim above will help us to show that the following construction is indeed a Segal space.

Construction of $\mathrm{PreCob}(n)$.

Let V be a finite-dimensional $\mathbb{R}$-vectorspace, $k \in \mathbb{N}_0$.

Define as a set $\mathrm{PreCob}(n)_k^V := \{ (M, t_0 \leq \ldots \leq t_k) \mid t_i \in \mathbb{R}\ \forall i,\ M \subseteq V \times \mathbb{R}$ an n-dim closed embedded submfld such that the natural projection $\pi : M \to \mathbb{R}$ is proper and $\mathrm{Crit}(\pi) \cap \{t_0, \ldots, t_k\} = \emptyset \}$

We can equip this set with a topology in the following way: Consider $\mathrm{Emb}(X, V\times\mathbb{R})$ with the Whitney topology, where $X \subseteq V \times \mathbb{R}$ is a mfld as in $\mathrm{PreCob}(n)_k^V$. Let $W \subset \mathrm{Emb}(X, V\times\mathbb{R})$ be open and $K \subset V\times\mathbb{R}$ compact. We can define the topology on $\mathrm{PreCob}(n)_k^V$ throught the neighbourhood bases of all possible manifolds, X, which are given by:

$$\mathcal{U}_{K,W} := \{ Y \subseteq V\times\mathbb{R} \mid Y \text{ an } n\text{-dim closed embedded submfld of } V\times\mathbb{R} \text{ such that } \exists f \in W : K \cap Y = K \cap f(X) \}$$

These neighbourhood bases together with the standart topology on $\mathbb{R}^{k+1}$ make $\mathrm{PreCob}(n)_k^V$ a top. space. (see picture at the end)

The morphism of Δ are sent to continuous maps under $\mathrm{PreCob}(n)_\bullet^V$ in the same way as $\mathrm{SemiCob}(n)_\bullet^V$ did it, only we don't have to intersect M with $V \times [t_{\theta(0)}, t_{\theta(k)}]$ anymore.

Together this gives us a simplicial space.

Again we choose an ∞-dimensional real vectorspace $\mathbb{R}^\infty$ and define

$$\mathrm{PreCob}(n)_k := \lim_{\substack{\longrightarrow \\ V \subset \mathbb{R}^\infty \text{ finite} \\ \text{dimensional}}} \mathrm{PreCob}(n)_k^V .$$

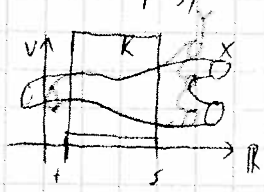
Claim: $\mathrm{PreCob}(n)_\bullet$ is a Segal space

This claim follows from the claim that $\mathrm{SemiCob}(n)_\bullet$ is a semiSegal space.

This space is the space we were looking for. However there is a small issue.

In general $\mathrm{PreCob}(n)_\bullet$ is not a complete Segal space (For $n \geq 6$ this follows from the s-cobordism theorem and the existence of certain Whitehead torsions). We can correct this by defining $\mathrm{Cob}_t(n)$ to be the completion of $\mathrm{PreCob}(n)_\bullet$ which always exist.
However the space $\mathrm{PreCob}(n)_\bullet$ contains more refined data which is useful for us. So we will work with $\mathrm{PreCob}(n)_\bullet$, while the $(\infty,1)$-category in the background is in fact $\mathrm{Cob}_t(n)$.

Picture of the topology on the space of manifolds occurring in $\mathrm{PreCob}(n)_\bullet^V$:



For all $Y \subseteq V \times \mathbb{R}$ closed embedded submanifolds like in $\mathrm{PreCob}(n)_\bullet^V$ such that $Y \cap K = f(X) \cap K$ for all $f \in W \subset \mathrm{Emb}(X, V \times \mathbb{R})$ we find a continuous path from X to Y by first going from X to $f(X)$ and then "sending the difference between $f(X)$ and Y outside of K to infinity".

References

[1] J. Lurie, On the classification of topological field theories, Current developments in mathematics, 2008, 129-280, Int. Press, Somerville, MA, 2009.