n-FOLD SEGAL SPACES

1. A review of iterated Segal spaces

1.1. The homotopy hypothesis.

Convention 1.1. A 0-*fold Segal space* is a topological space.

The *homotopy hypothesis* states that topological spaces are models for ∞-groupoids (also refered to as $(\infty,0)$-categories). Given a topological space X, points are thought as objects, paths between them as 1-morphisms, homotopies between paths as 2-morphisms, homotopies between homotopies as 3-morphisms, ... it is clear that all n-morphisms are invertible (up to homotopy, i.e. higher morphisms).

1.2. Segal spaces.

Definition 1.2. A (1-*fold*) *Segal space* is a simplicial space $X = X_\bullet$ such that for any $n, m \geqslant 0$ the map

$$X_{m+n} \longrightarrow X_m \overset{h}{\underset{X_0}{\times}} X_n$$

is a weak equivalence.

Observe that following [1] we drop the Reedy fibrant condition which guarantees in particular that the canonical map

$$X_m \underset{X_0}{\times} X_n \longrightarrow X_m \overset{h}{\underset{X_0}{\times}} X_n$$

is a weak equivalence.

Example 1.3. Let $\mathcal{C}$ be a small topological category (i.e. a small category enriched over topological spaces). Then we have a Segal space $X^{\mathcal{C}} := N(\mathcal{C})$.

The above example motivates the following interpretation of Segal spaces as models for $(\infty,1)$-categories: if $X_\bullet$ is a Segal space then we view $\widehat{X_0}$ (meaning the set underlying the space X_0) as the set of objects, X_1 as the $(\infty,0)$-category of arrows, and more generally X_n as the $(\infty,0)$-category of n-tuples of composable arrows.

1.3. The homotopy category of a Segal space.

To any higher category one can intuitively associate an ordinary category having the same objects, with morphisms being 2-isomorphism classes of 1-morphisms.

The *homotopy category* $\mathrm{h}_1(X)$ of a Segal space $X = X_\bullet$ is defined as follows: it has $\widehat{X_0}$ as set of objects and

$$\mathrm{Hom}_{\mathrm{h}_1(X)}(x,y) := \pi_0\left(\mathrm{Hom}_X(x,y)\right) = \pi_0\left(\{x\} \overset{h}{\underset{X_0}{\times}} X_1 \overset{h}{\underset{X_0}{\times}} \{y\}\right).$$

The composition of morphisms is defined as follows:

$$\begin{aligned}\left(\{x\}\overset{h}{\underset{X_0}{\times}}X_1\overset{h}{\underset{X_0}{\times}}\{y\}\right)\times\left(\{y\}\overset{h}{\underset{X_0}{\times}}X_1\overset{h}{\underset{X_0}{\times}}\{z\}\right) &\longrightarrow \{x\}\overset{h}{\underset{X_0}{\times}}X_1\overset{h}{\underset{X_0}{\times}}X_1\overset{h}{\underset{X_0}{\times}}\{z\}\\ &\overset{\sim}{\longleftarrow} \{x\}\overset{h}{\underset{X_0}{\times}}X_2\overset{h}{\underset{X_0}{\times}}\{z\}\\ &\longrightarrow \{x\}\overset{h}{\underset{X_0}{\times}}X_1\overset{h}{\underset{X_0}{\times}}\{z\}\,.\end{aligned}$$

The second arrow happens to go in the wrong way but it is a weak equivalence, and thus it induces an isomorphism on π_0.

1.4. **Complete Segal spaces.**

In our definition of the homotopy category $\mathrm{h}_1(X)$ of a Segal space $X = X_\bullet$, as well as in our interpretation of X as an $(\infty,1)$-category, we do not seem to use the information coming from the topology of X_0.

Loosely speaking, we would like to see the topology of X_0 as encoding the ∞-groupoid of invertible arrows in our $(\infty,1)$-category. We will say that a element $f \in X_1$ (an arrow) is *invertible* if its image through

$$\{x\}\underset{X_0}{\times}X_1\underset{X_0}{\times}\{y\}\longrightarrow\{x\}\overset{h}{\underset{X_0}{\times}}X_1\overset{h}{\underset{X_0}{\times}}\{y\}\longrightarrow\pi_0\left(\{x\}\overset{h}{\underset{X_0}{\times}}X_1\overset{h}{\underset{X_0}{\times}}\{y\}\right)=\mathrm{Hom}_{\mathrm{h}_1(X)}(x,y)\,,$$

where x and y are source and target (i.e. the two faces) of f, is an invertible morphism in $\mathrm{h}_1(X)$.

Let us denote by X_1^{inv} the subspace of invertible arrows, and observe that the map $X_0 \to X_1$ factors through X_1^{inv} (because the image of an $x \in X_0$ through $X_0 \to X_1 \to \mathrm{Hom}_{\mathrm{h}_1(X)}(x,x)$ is id_x).

Definition 1.4. A Segal space is *complete* if the map $X_0 \to X_1^{inv}$ is a weak equivalence. An $(\infty,1)$-*category* is a complete Segal space.

Segal spaces can always be completed, therefore we we will not put to much emphasis on the completeness condition unless this is strictly necessary.

1.5. *n*-**fold Segal spaces.**

Definition 1.5. An *n-fold Segal space* is an n-fold simplicial space $X = X_{\bullet,\dots,\bullet}$ such that

(i) For every $1 \leqslant i \leqslant n$, and every $k_1,\dots,k_{i-1},k_{i+1},\dots,k_n \geqslant 0$,

$$X_{k_1,\dots,k_{i-1},\bullet,k_{i+1},\dots,k_n}$$

is a Segal space.

(ii) For every $1 \leqslant i \leqslant n$, and every $k_1,\dots,k_{i-1} \geqslant 0$,

$$X_{k_1,\dots,k_{i-1},0,\bullet,\dots,\bullet}$$

is essentially constant[1].

[1] An m-fold simplicial space $X_{\bullet,\dots,\bullet}$ is essentially constant if there is a weak homotopy equivalence of m-fold simplicial spaces $Y \to X$, where Y is constant.

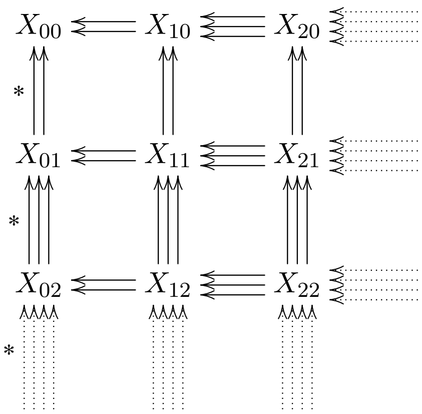


Diagram of the 2-fold Segal space. Asterisks indicate weak equivalences.

An n-fold Segal space is *complete*, if

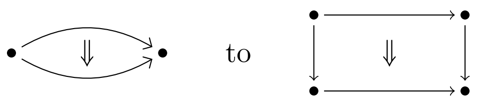
(iii) For every $0 \leqslant i \leqslant n$, $k_1, \ldots, k_{i-1} \geqslant 0$,

$$X_{k_1,\ldots,k_{i-1}}, \bullet, 0, \ldots, 0$$

is a complete Segal space.

We now explain how n-fold Segal spaces can be thought of as higher categories.

First of all, let's try to understand condition (ii) in the definition of an n-fold Segal space. For $n = 2$ one can think of this as "fattening" the objects in a bicategory:



where points are objects, i.e. elements in $X_{0,0}$, horizontal arrows are elements in $X_{1,0}$, which correspond to 1-morphisms, and the vertical arrows are elements in $X_{0,1}$, which are essentially just identities because of condition (ii). The arrow in the face should be thought of as a 2-morphism which is an element in $X_{1,1}$.

For $n = 3$, the same idea works. Elements of $X_{0,0,0}$ can be thought as being objects of the category, elements of $X_{1,1,0}$ as 1-morphisms and elements of $X_{1,1,1}$ as 2-morphisms.

In general, with higher morphisms, we should think of elements of $X_{0,\ldots,0}$ as the objects of our category, $X_{1,\ldots,1,0,\ldots,0}$ as i-th "horizontal" arrows (i is the number of 1's), which correspond to the i-morphisms. Moreover, the others are the "vertical" arrows, which are essentially just identities of lower morphisms.

Here is how the diagram of a 3-fold Segal Space looks like.

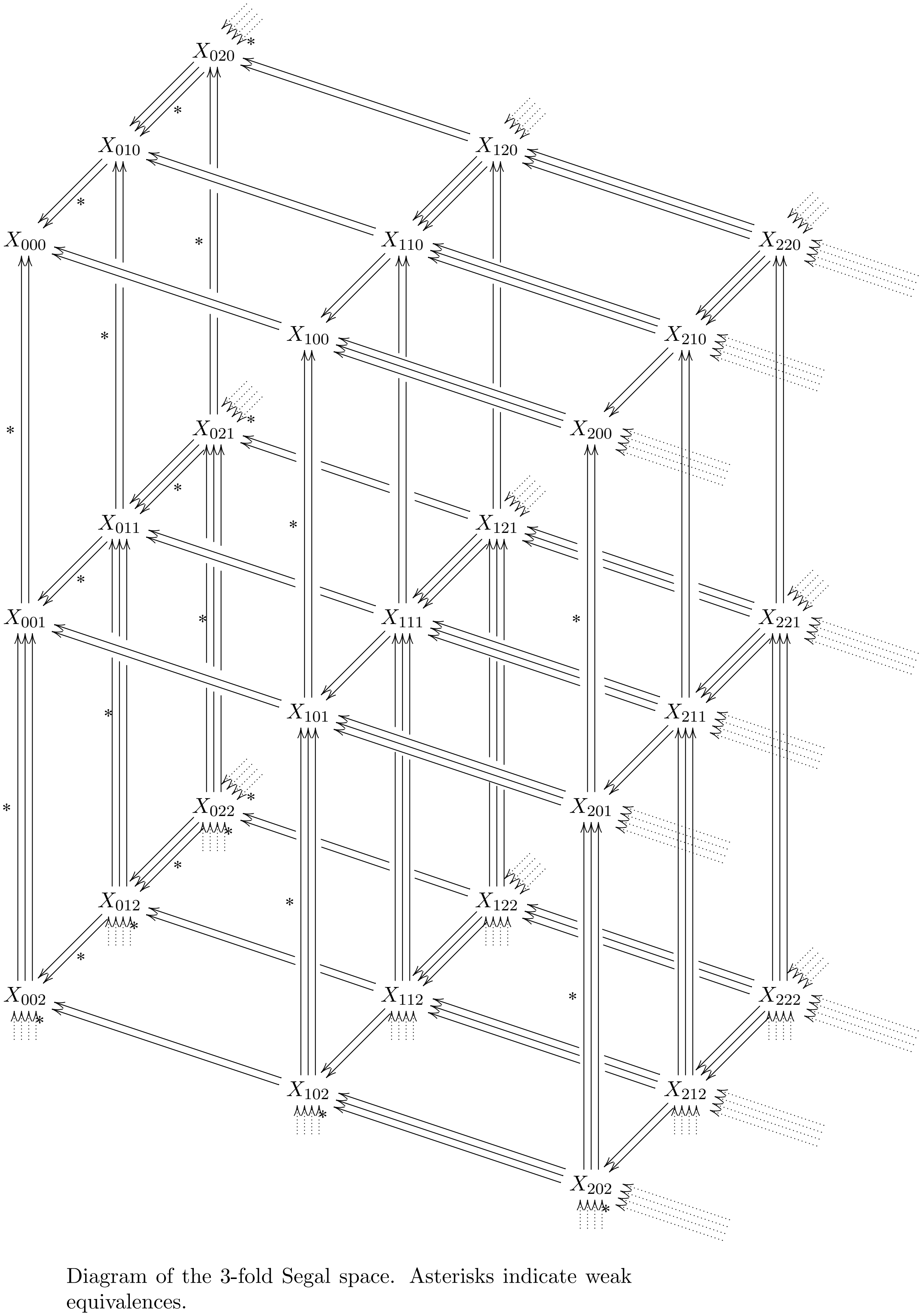


Diagram of the 3-fold Segal space. Asterisks indicate weak equivalences.

We will now describe several constructions that someone would like to be able to do when dealing with models of higher categories.

1.6. **Truncation.** Given an (∞, n)-category, its (∞, k)-truncation ($k \leqslant n$) is the (∞, k)-category obtained from it by discarding the non-invertible m-morphisms for $k < m \leqslant n$.

In terms of iterated Segal spaces, if $X = X_{\bullet,\dots,\bullet}$ is an n-fold Segal space then its (∞, k)-truncation is the k-fold Segal space

$$\tau_k X = X_{\underbrace{\bullet,\dots,\bullet}_{k \text{ times}},\underbrace{0,\dots,0}_{n-k \text{ times}}} \cdot$$

Note that if X is complete then so is $\tau_k X$ by condition (3).

For example, in the diagram of the 2-fold Segal space, $\tau_1 X$ is the upper line - we lose all informations regarding the 2-morphisms. In the diagram of the 3-fold Segal space, $\tau_1 X$ is the upper face of the cube and $\tau_2 X$ is the edge containing $X_{0,0,0}, X_{1,0,0}, X_{2,0,0}, \dots$

Warning 1.6. Truncation does not behave well with completion. We should ALWAYS complete an n-fold Segal space before truncating it, otherwise the categorical interpretation of truncation fails: e.g. if $X = X_\bullet$ is a (1-fold) Segal space then X_0 does not identify with the ∞-groupoid of invertible morphisms unless X is complete.

1.7. **Extension.** Any (∞, n)-category can be viewed as an $(\infty, n+1)$-category. In terms of iterated Segal spaces, any n-fold Segal space can be viewed as a constant $(n + 1)$-fold Segal space. We call ε the extension functor, which is left adjoint to τ_1. Moreover, the unit $\mathrm{id} \to \tau_1 \circ \varepsilon$ of the adjunction is the identity.

1.8. **Iterated Segal spaces of morphisms.** Given two objects x, y in an (∞, n)-category we would like to have an $(\infty, n-1)$-category of morphisms from x to y.

Let $X = X_{\bullet,\cdots,\bullet}$ be an n-fold Segal space. As we have seen above one should think of objects as elements in the set $\widehat{X_{0,\dots,0}}$. The $(n-1)$-fold Segal space of morphisms between two given objects $x, y \in \widehat{X_{0,\dots,0}}$ is

$$\mathrm{Hom}_X(x,y)_{\bullet,\cdots,\bullet} := \{x\} \underset{X_{0,\bullet,\cdots,\bullet}}{\overset{h}{\times}} X_{1,\bullet,\cdots,\bullet} \underset{X_{0,\bullet,\cdots,\bullet}}{\overset{h}{\times}} \{y\}\,.$$

Sanity check 1.7 (Compatibility with extension)**.** Let X be a topological space (an $(\infty, 0)$-category), viewed as a constant Segal space $X_\bullet$ (an $(\infty, 1)$-category). For any two objects $x, y \in X_0 = X$ the topological space (i.e. $(\infty, 0)$-category) of morphisms from x to y is

$$\mathrm{Hom}_{X_\bullet}(x,y) := \{x\} \underset{X_0}{\overset{h}{\times}} X_1 \underset{X_0}{\overset{h}{\times}} \{y\} = \{x\} \underset{X}{\overset{h}{\times}} \{y\} = \mathrm{Path}_X(x,y)\,,$$

as expected.

1.9. **The homotopy bicategory of a 2-fold Segal space.** To any higher category one can intuitively associate a bicategory having the same objects and 1-morphisms, and with 2-morphisms being 3-isomorphism classes of the original 2-morphisms.

The *homotopy bicategory* $\mathrm{h}_2(X)$ of a 2-fold Segal space $X = X_{\bullet,\bullet}$ is defined as follows: it has $\widehat{X_{0,0}}$ as set of objects, and

$$\mathrm{Hom}_{\mathrm{h}_2(X)}(x,y) = \mathrm{h}_1\big(\mathrm{Hom}_X(x,y)\big) = \mathrm{h}_1\left(\{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{y\}\right)$$

as Hom categories. The horizontal composition is defined as follows:

$$\begin{aligned}\left(\{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{y\}\right) \times \left(\{y\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\}\right) &\longrightarrow \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\} \\ &\overset{\sim}{\longleftarrow} \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{2,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\} \\ &\longrightarrow \{x\} \overset{h}{\underset{X_{0,\bullet}}{\times}} X_{1,\bullet} \overset{h}{\underset{X_{0,\bullet}}{\times}} \{z\}\,.\end{aligned}$$

The second arrow happens to go in the wrong way but it is a weak equivalence. Therefore after taking h_1 it turns out to be an equivalence of categories, and thus to have an inverse (assuming the axiom of choice).

References

[1] J. Lurie, On the Classification of Topological Field Theories, Current developments in mathempatics, 2008, 129–280, Int. Press, Sommerville, MA, 2009. Available at `http://www.math.harvard.edu/~lurie/papers/cobordism.pdf`.