Monoidal Structures on Higher Categories

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1 Monoidal Structures on Simplicial Categories

Let $\mathscr{C}$ be a simplicial category, that is a category enriched over simplicial sets. Such categories are a model for $(\infty, 1)$-categories. For them, one can define monoidal structures by "enriching" the definition of a monoidal structure on a category:

Definition 1.1. A (symmetric) monoidal structure on $\mathscr{C}$ consists of a functor (of enriched categories) $\otimes\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, an object $\mathbb{1} \in \mathscr{C}$ together with natural isomorphisms

$$
\begin{aligned}
(X \otimes Y) \otimes Z &\cong X \otimes (Y \otimes Z) \\
X \otimes \mathbb{1} &\cong X \\
\mathbb{1} \otimes X &\cong X \\
(X \otimes Y &\cong Y \otimes X)
\end{aligned}
$$

for all $X, Y, Z \in \mathscr{C}$ which satisfy the same coherence axioms as the analogous isomorphisms in a usual (symmetric) monoidal category.

As an example we consider the category $\mathscr{C}\mathscr{C}$ of chain complexes over a field k. First we define a simplicial enrichment: For $n \geq 0$ let R_n be the k-algebra $k[x_0, \ldots, x_n]/(\sum_{i=0}^{n} x_i - 1)$. This should be thought of as the algebraic n-simplex over k. For varying n these algebras form a simplicial object by associating to a map $\rho\colon [n] \to [m]$ the homomorphism $R_m \to R_n$ given by $x_i \mapsto \sum_{j\colon \rho(j)=i} x_j$. For $n \geq 0$ let $\Omega^*_{\Delta^n}$ be the algebraic de Rham complex of R_n over k:

$$0 \to R_n \to \Omega^1_{R_n/k} \to \Omega^2_{R_n/k} \to \ldots \to \Omega^n_{R_n/k} \to 0$$

By functoriality of the algebraic de Rham complex for varying n these form a simplicial object. Now for any $X, Y \in \mathscr{C}\mathscr{C}$ we let $\mathrm{Hom}^{\Delta}_{\mathscr{C}\mathscr{C}}(X, Y)_n := \mathrm{Hom}_{\mathscr{C}\mathscr{C}}(X, Y \otimes \Omega^*_{\Delta^n})$. By functoriality these sets form a simplicial set $\mathrm{Hom}^{\Delta}(X, Y)$. We define a simplicial category $\mathscr{C}\mathscr{C}^{\Delta}$ which has the same objects as $\mathscr{C}\mathscr{C}$ and these simplicial sets as hom objects. This definition is motived by the following fact:

Claim 1.2. *For any $X, Y \in \mathscr{C}\mathscr{C}$, elements of $\mathrm{Hom}^{\Delta}_{\mathscr{C}\mathscr{C}}(X, Y)_1$ correspond to homotopies between two chain maps $X \to Y$.*

Proof. We identify R_1 with $k[x]$ via $x \mapsto x_1$. Then $\Omega^*_{\Delta^1}$ can be written as

$$0 \to k[x] \to k[x]\mathrm{d}x \to 0.$$

Consequently, using the notation $X[x] := X \otimes k[x]$ (with $k[x]$ concentrated in degree 0), a chain map $h\colon X \to Y \otimes \Omega^*_{\Delta^1}$ is given by a map $f\colon X \to Y[x]$ of degree 0 and a map $X \to Y[x]\mathrm{d}x$ which we write as $g(x)\mathrm{d}x$ with $g\colon X \to Y[x]$ of degree -1. Using the fact that the differential on $Y \otimes \Omega^*_{\Delta^1}$ is given by $d_{Y\otimes\Omega^*_{\Delta^1}} := d_Y \otimes \mathrm{id} + id \otimes d_{dR}$, where d_{dR} is the de Rham differential, the condition $d_{Y\otimes\Omega^*_{\Delta^1}} h = h d_x$ is equivalent to $f'\mathrm{d}x + d_Y f - d_Y g \mathrm{d}x = f d_X + g d_X \mathrm{d}x$, where f' denotes the formal differential of f with respect to x. By splitting this up into graded pieces we get the following two conditions:

$$d_Y f = f d_X \tag{1.3}$$

$$f' = g d_X + d_Y g \tag{1.4}$$

The first condition means that f is a family of chain maps, and we show now that the second condition gives us a homotopy between $f(0)$ and $f(1)$: There is a formal k-linear integration map $\int_0^1 \colon k[x] \to k$ which sends x^n to $x^{n+1}/(n+1)|_0^1 = 1/(n+1)$. By tensoring with the identity on Y we get a map $\int_0^1 \colon Y[x] \to Y$. Applying this operator to (1.4) yields

$$f(1) - f(0) = (\int_0^1 g) d_X + d_Y(\int_0^1 g).$$

Thus any h as above yields a homotopy between $f(0)$ and $f(1)$.

On the other hand, assume that we have chain maps $f_0, f_1 \colon X \to Y$ as well as a homotopy $g \colon X \to Y$ (of degree 1) such that $f_1 - f_0 = g d_X + d_Y g$. Then if we define $f := f_0(1-x) + f_1 x$ and $h := f + g\mathrm{d}x$ we see that conditions (1.3) and (1.4) are satisfied, so that we obtain $h \in \mathrm{Hom}^\Delta(X,Y)_1$. This shows the claim. □

Similarly one can see that elements of $\mathrm{Hom}^\Delta_{\mathscr{C}\mathscr{C}}(X,Y)_n$ for $n \geq 2$ correspond to homotopies between homotopies between ... between chain maps.

Now we can define a monoidal structure on $\mathscr{C}\mathscr{C}^\Delta$ as follows: On objects we take the usual tensor product $\mathscr{C}\mathscr{C} \otimes \mathscr{C}\mathscr{C} \to \mathscr{C}\mathscr{C}$. On morphism, we act by the composition

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{C}\mathscr{C}}(X, Y \otimes \Omega^*_{\Delta^n}) \times \mathrm{Hom}_{\mathscr{C}\mathscr{C}}(Z, W \otimes \Omega^*_{\Delta^n}) \\
&\qquad \to \mathrm{Hom}_{\mathscr{C}\mathscr{C}}(X \otimes Z, Y \otimes W \otimes \Omega^*_{\Delta^n} \otimes \Omega^*_{\Delta^n}) \\
&\qquad\qquad \to \mathrm{Hom}_{\mathscr{C}\mathscr{C}}(X \otimes Z, Y \otimes W \otimes \Omega^*_{\Delta^n})
\end{aligned}$$

where the first map is given by the usual tensor product in $\mathscr{C}\mathscr{C}$ and the second map is induced by the wedge product map $\Omega^*_{\Delta^n} \otimes \Omega^*_{\Delta^n} \to \Omega^*_{\Delta^n}$.

2 Monoidal Structures on n-fold Segal Spaces

Giving a monoidal category is the same as a 2-category with a single object: If one has the latter, the endomorphism of the unique object form a category on which one has the additional structure given by the composition of endomorphisms. This structure amounts to a monoidal structure on the category. This motivates the following definition:

Definition 2.1. A *monoidal n-fold Segal space* is a $(n+1)$-fold Segal space $X_{\bullet,\dots,\bullet}$ such that the space $X_{0,\dots,0}$ is connected.

Recall:

Definition 2.2. Let $X_{\bullet,\dots,\bullet}$ be an n-fold Segal space. For "objects" $x, y \in X_{0,\dots,0}$, the $(n-1)$-fold Segal space of morphisms from x to y is given by

$$\mathrm{Hom}_X(x,y)_{\bullet,\dots,\bullet} := \{x\} \times^h_{X_{0,\bullet,\dots,\bullet}} X_{1,\bullet,\dots,\bullet} \times^h_{X_{0,\bullet,\dots,\bullet}} \{y\}.$$

Then the underlying n-fold Segal space of a monoidal n-fold Segal space X is the space $\mathrm{Hom}_X(*,*)$ for any $* \in X_{0,\ldots,0}$.

To define symmetric monoidal Segal spaces, we push the above idea further: Instead of considering higher categories with just a single object, we consider higher categories with a single object, a single morphism, a single 2-morphism and so on up to some level k. The following example gives an indication that this is the right idea:

Example 2.3. Recall that a monoid is the same as a category with a single object. Going one level higher, a commutative monoid is the same as a 2-category with a single object and a single morphism: Such a category is determined by $G := \mathrm{Hom}(\mathrm{id}_*, \mathrm{id}_*)$, where $*$ is the unique object. On G we have two different monoid structures $G \times G \to G$ given by horizontal and vertical composition. They are compatible in the sense that if one considers G as a monoid through one of these two maps, the map $G \times G \to G$ giving the other multiplication is a homomorphism. Thus the Eckmann-Hilton argument implies that these two monoid structures coincide and are commutative.

In general, it is not known whether going up to some level k as above suffices to give a symmetric monoidal structure on a (∞, n)-category. Thus we go all the way up to infinity:

Definition 2.4. For $k \geq 0$, a *k-tuply monoidal n-fold Segal space* is a $(n+k)$-fold Segal space X such that for all $0 \leq \ell \leq k-1$ the space $X_{1,\ldots,1,0,\ldots 0}$ (with exactly ℓ ones) is connected.

A *symmetric monoidal n-fold Segal space* consists of a sequence $(X^k)_{k\geq 0}$ with X^k a k-tuply monoidal n-fold Segal space together with weak equivalences $X^{k-1} \simeq \mathrm{Hom}_{X^k}(*,*)$ for objects $* \in X^k_{0,\ldots,0}$.

Now we show how to make the bordism categories into symmetric monoidal spaces in this sense. Recall:

Definition 2.5.
- Let V be a finite-dimensional vector space. Let $n \geq 0$ and $l \leq n$. For every n-tuple $k_1, \ldots, k_n \geq 0$, we let $(\mathrm{PBord}_n^{l,V})_{k_1,\ldots,k_{n-l}}$ be the collection of tuples $(M \hookrightarrow V \times \mathbb{R}^{n-l}, t_0^i \leq \ldots \leq t_{k_i}^i)_{i=1,\ldots,n-l})$ satisfying
  1. M is a closed n-dimensional submanifold of $V \times \mathbb{R}^{n-l}$,
  2. the composition $\pi : M \hookrightarrow V \times \mathbb{R}^{n-l} \twoheadrightarrow \mathbb{R}^{n-l}$ is a proper map,
  3. for every $S \subseteq \{1, \ldots, n-l\}$ and for every collection $\{j_i\}_{i\in S}$, where $0 \leq j_i \leq k_i$, the composition $p_S : M \xrightarrow{\pi} \mathbb{R}^{n-l} \to \mathbb{R}^S$ does not have $(t_{j_i})_{i\in S}$ as a critical value[1].
  4. for every $x \in M$ such that $p_{\{i\}}(x) \in \{t_0^i, \ldots, t_n^i\}$, the map $p_{\{i+1,\ldots,n-l\}}$ is submersive at x.
- $\mathrm{PBord}_n^l = \varinjlim_V \mathrm{PBord}_n^{l,V}$

Remark 2.6. This defines an $(\infty, n-l)$-category and is well-defined even for negative l. Note that l corresponds to the "level" we're extending down to, so $\mathrm{PBord}_n^0 = \mathrm{PBord}_n$.

We will omit the vector space V in our notation from now on and just remember that in the end we need to take the limit over vector spaces.

Let $\emptyset := (\emptyset \hookrightarrow V, 0, \ldots, 0) \in (\mathrm{PBord}_n^{-1})_{0,\ldots,0}$.

[1]By critical value we mean that $\mathrm{d}p_S$ is surjective at the preimages of $(t_{j_i})_{i\in S}$.

Claim 2.7.
$$\mathrm{Hom}_{\mathrm{PBord}_n^{-1}}(\emptyset,\emptyset) \xrightarrow{\simeq} \mathrm{PBord}_n\,.$$

Sketch of proof. The map sends an element in $\mathrm{Hom}_{\mathrm{PBord}_n^{-1}}(\emptyset,\emptyset)_{k_1,\dots,k_n}$ represented by

$$\big(M \hookrightarrow V\times(\mathbb{R}\times\mathbb{R}^n), s_0\leq s_1, (t_0^i\leq\cdots\leq t_{k_i}^i)_{i=1,\dots,n}\big) \in (\mathrm{PBord}_n^{-1})_{1,k_1,\dots,k_n}$$

to

$$(M\hookrightarrow \underbrace{(V\times\mathbb{R})}_{=\tilde V}\times\mathbb{R}^n, (t_0^i\leq\cdots\leq t_{k_i}^i)_{i=1,\dots,n}) \in (\mathrm{PBord}_n)_{k_1,\dots,k_n}.$$

Conversely, we can send

$$(M\hookrightarrow V\times\mathbb{R}^n, (t_0^i\leq\cdots\leq t_{k_i}^i)_{i=1,\dots,n})$$

to

$$(M\hookrightarrow V\times(\mathbb{R}\times\mathbb{R}^n), s_0<s_1, (t_0^i\leq\cdots\leq t_{k_i}^i)_{i=1,\dots,n})$$

by the inclusion $i:\mathbb{R}^n\hookrightarrow\mathbb{R}^{n+1}$, $(x_1,\dots,x_n)\mapsto(s,x_0,\dots,x_n)$, where $s_0<s<s_1$. This depends on the choice of $s_0,s_1\in(\mathbb{R}_+)^2$ and s, but the set of choices is contractible. One can check easily that the maps are well-defined (up to a contractible choice in the second case). □

This endows PBord_n with the structure of a monoidal n-fold Segal space. This procedure can be iterated: Similarly the the above one has $\mathrm{PBord}_n^l \simeq \mathrm{Hom}_{\mathrm{PBord}_n^{l+1}}(\emptyset,\emptyset)$ for $l\leq 0$. This endows PBord_n with the structure of a symmetric monoidal n-fold Segal space.

Claim 2.8. *The symmetric monoidal structure on* PBord_1 *endows* $1\,\mathrm{Cob}$*, which is equivalent to its homotopy category* $\mathrm{h}_1(\mathrm{Bord}_1^{-1})$*, with the symmetric monoidal structure given by disjoint union.*

Proof. Consider two objects or morphisms M and N in $1\,\mathrm{Cob}$. They come from elements

$$M := (M\hookrightarrow V\times\mathbb{R}, t_0\leq\cdots\leq t_k)$$

$$N := (N\hookrightarrow V\times\mathbb{R}, \tilde{t_0}\leq\cdots\leq\tilde{t_k})$$

in $(\mathrm{Bord}_1)_k$ (for $k=0$ or 1). Fixing real numbers $s_0<s<s_1$, they are sent to elements

$$i(M) := (M\hookrightarrow V\times\mathbb{R}\times\mathbb{R}, s_0\leq s_1, t_0\leq\cdots\leq t_k)$$

$$i(N) := (N\hookrightarrow V\times\mathbb{R}\times\mathbb{R}, s_0\leq s_1, \tilde{t_0}\leq\cdots\leq\tilde{t_k})$$

in $(\mathrm{Hom}_{\mathrm{Bord}_1^{-1}}(\emptyset,\emptyset))_k$. Recall the "composition law" in a 2-fold Segal space X:

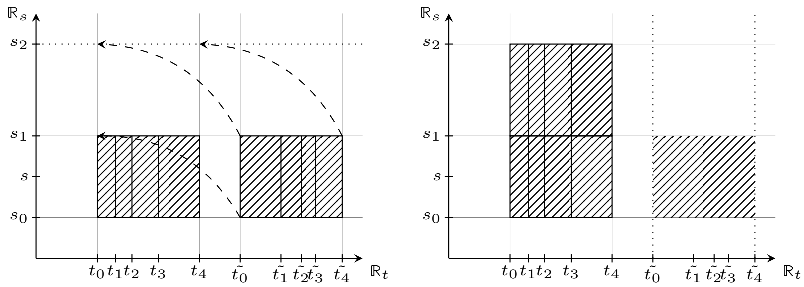
$$\begin{aligned}\Big(\{x\}\overset{h}{\underset{X_{0,\bullet}}{\times}}X_{1,\bullet}\overset{h}{\underset{X_{0,\bullet}}{\times}}\{y\}\Big)\times\Big(\{y\}\overset{h}{\underset{X_{0,\bullet}}{\times}}X_{1,\bullet}\overset{h}{\underset{X_{0,\bullet}}{\times}}\{z\}\Big) &\longrightarrow \{x\}\overset{h}{\underset{X_{0,\bullet}}{\times}}X_{1,\bullet}\overset{h}{\underset{X_{0,\bullet}}{\times}}X_{1,\bullet}\overset{h}{\underset{X_{0,\bullet}}{\times}}\{z\}\\ &\xleftarrow{\sim} \{x\}\overset{h}{\underset{X_{0,\bullet}}{\times}}X_{2,\bullet}\overset{h}{\underset{X_{0,\bullet}}{\times}}\{z\}\\ &\longrightarrow \{x\}\overset{h}{\underset{X_{0,\bullet}}{\times}}X_{1,\bullet}\overset{h}{\underset{X_{0,\bullet}}{\times}}\{z\}\,.\end{aligned}$$

Here the second arrow goes the wrong way but it is a weak equivalence.

We choose a differentiable path in $(\mathrm{Bord}_1^{-1})_{1,k}$ from the second element to an element

$$(N' \hookrightarrow V \times \mathbb{R} \times \mathbb{R}, s_1 \leq s_2, t_0 \leq \cdots \leq t_k),$$

by moving N as illustrated in the following drawings (i.e. we're just changing the embedding of N into $V \times \mathbb{R} \times \mathbb{R}$):



Then $(i(M), i(N))$ is an element in

$$\left(\{\emptyset\} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} (\mathrm{Bord}_1^{-1})_{1,k} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} \{\emptyset\}\right) \times \left(\{\emptyset\} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} (\mathrm{Bord}_1^{-1})_{1,k} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} \{\emptyset\}\right)$$

which lies in the image of

$$\left(\{\emptyset\} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} (\mathrm{Bord}_1^{-1})_{2,k} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} \{\emptyset\}\right)$$

and therefore can be composed to an element of

$$\left(\{\emptyset\} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} (\mathrm{Bord}_1^{-1})_{1,k} \overset{h}{\underset{(\mathrm{Bord}_1^{-1})_{0,k}}{\times}} \{\emptyset\}\right)$$

by forgetting s_1.

By going back under the bijection of Claim 2.7 and going to the homotopy category of PBord_1^{-1} we get the tensor product of M and N. This amounts to forgetting s_0 and s_1 and taking the fiber over t_0 (resp. $[t_0, t_1]$). Thus we see that this tensor product is the disjoint union of M and N. $\square$