

Around Fully dualizable objects in (∞, n) -categories

— an introduction

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Bimod := \otimes -bicategory w/ objects: k -algebras (assoc., unital) & fixed ring

1-morphisms: bimodules

$$A \xrightarrow{M} B,$$

M a left B -module
right A -module

Notation: $A \xrightarrow{{}_B M_A} B$

2-morphisms: maps of bimodules

\otimes -structure: $A \otimes B$ is $A \otimes B$ as algebras

$$A \otimes A' \xrightarrow{\begin{matrix} M_A & \otimes & M_{A'} \\ B & & B' \end{matrix}} B \otimes B'$$

composition:

$$A \xrightarrow{{}_B M_A} B \xrightarrow{{}_C N_B} C$$

$\xrightarrow{{}_C N_B \otimes {}_B M_A}$

CatMod := bicategory w/

objects: algebras A thought of as A -mod
" "
cat. of left A -modules

1-morphisms: $A\text{-mod} \xrightarrow{F} B\text{-mod}$
cocont. functors
(commute w/ colimits)

Remarks: CatMod and $B\text{-mod}$ are presentable
categories $\Rightarrow F: A\text{-mod} \rightarrow B\text{-mod}$ is
left adjoint iff F is cocontinuous
(general adjoint functor theorem).

② All cocontinuous $F: A\text{-mod} \rightarrow B\text{-mod}$

are of the type

$$X \longmapsto M_{B A} \otimes_A X,$$

where $M_{B A} := F(A)$

$$A \longmapsto M_{B A} \otimes_A A = M_{B A}$$

2-morphisms: natural transformations of functors
 \Leftrightarrow maps of bimodules

There is an equivalence of ^{monoidal} bicategories

$$\text{Bimod} \xrightarrow{\sim} \text{CatMod}$$

\otimes -structure on CatMod

$$A\text{-mod} \otimes B\text{-mod} := A \otimes B\text{-mod}$$

Now to every bimodule we associate an adjunction

$$A\text{-mod} \begin{array}{c} \xrightarrow{M \otimes_A -} \\ \xleftarrow{\text{Hom}_B(M, -)} \end{array} B\text{-mod}$$

$\forall X \in A\text{-mod}, Y \in B\text{-mod}$

$$\frac{M \otimes_A X \longrightarrow Y \quad \text{in } B\text{-mod}}{X \longrightarrow \text{Hom}_B(M, Y) \quad \text{in } A\text{-mod}}$$

(i.e. there is a bijection btw. the set of the maps on top and the set of the maps on the bottom.)

Question: When does a bimodule $M_{B A}$ have a left or right adjoint in Bimod?

Answer: $M_{B A}$ has a right adjoint if it is projective of finite type / B

$M_{B A}$ has a left adjoint " / A "

right adjoint:

The question reduces to:

When is $\text{Hom}_B(M, -)$ isomorphic to $N_{A B} \otimes_B -$?

If such an iso exists, $N = \text{Hom}_B(M, B)$!

Thus, we want to know when

$$\text{Hom}_B(M, Y) \cong \text{Hom}_B(M, B) \otimes_B Y \quad ?$$

A: When M is projective of finite type / B.

left adjoint:

$$\begin{array}{ccc}
 & \xleftarrow{N \otimes_B -} & \\
 A\text{-mod} & \xrightarrow{M \otimes_A -} & B\text{-mod} \\
 & \xleftarrow{\text{Hom}_B(M, B) \otimes_B -} &
 \end{array}$$

\exists left adj. if $\exists N$ st. $\text{Hom}_A(N, A) \cong M$.

$N = \text{Hom}_A(M, A)$ is OK if M proj. of ft. / A.

$$\begin{array}{ccc}
 & \text{Hom}_B(\text{Hom}_A(M, A), B) & \\
 & \xrightarrow{\hspace{10em}} & \\
 & \xleftarrow{\text{Hom}_A(M, A)} & \\
 A\text{-mod} & \xrightarrow{M} & B\text{-mod} \\
 & \xleftarrow{\text{Hom}_B(M, B)} & \\
 & \xrightarrow{\text{Hom}_A(\text{Hom}_B(M, B))} & \\
 & \vdots &
 \end{array}$$

(if they exist.)

Question: If M is projective of finite type over A and B , does this imply that all duals are projective of finite type over A and B ?

dualizable objects in Bimod?

Recall: The dual of A is A° .

$$\text{ev}: A^\circ \otimes A \xrightarrow{e_{A^\circ \otimes A}} k$$

$$\text{coev}: k \xrightarrow{c_{A^\circ \otimes A}} A^\circ \otimes A^\circ$$

A is a left $A^\circ \otimes A^\circ$ -module: $(A^\circ \otimes A^\circ) \otimes A \longrightarrow A$
 $\alpha \otimes \beta \cdot a \longmapsto \alpha a \beta$

$$\begin{array}{ccc}
 \text{ev}: & A^\circ\text{-mod} \otimes A\text{-mod} & \longrightarrow k\text{-mod} \\
 & \parallel & \\
 & A^\circ \otimes A\text{-mod} & \xrightarrow{A^\circ \otimes -} \\
 & & A^\circ \otimes A
 \end{array}$$

$$X \longmapsto A^\circ \otimes_{A^\circ \otimes A} X = X / [A, X]$$

$$= X / \alpha x \sim x \alpha$$

"coinvariant of X "

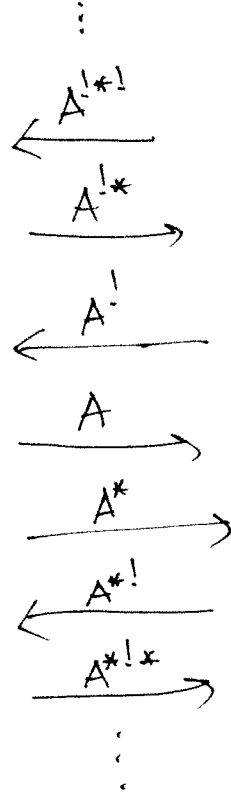
$$\begin{array}{ccc} \text{coev:} & \mathbb{k}\text{-mod} & \longrightarrow A \otimes A^\circ\text{-mod} \\ & \Upsilon & \longmapsto A \otimes \Upsilon \end{array}$$

- 1) ev has a right adjoint if A is projective of finite type / \mathbb{k}
- 2) " left " " / $A^\circ \otimes A$
- 3) coev " right " / $A^\circ \otimes A^\circ$
- 4) " left " / \mathbb{k}

Note: 1) \Leftrightarrow 4) 3) \Leftrightarrow 2) \Leftrightarrow A separable
 \Leftrightarrow A finite dimension

Question: Is finite dimensional and separable enough to have the full hierarchy of adjoints?

$A^\circ\text{-Mod} \otimes A\text{-Mod}$



$k\text{-Mod}$

Where $M^! =$

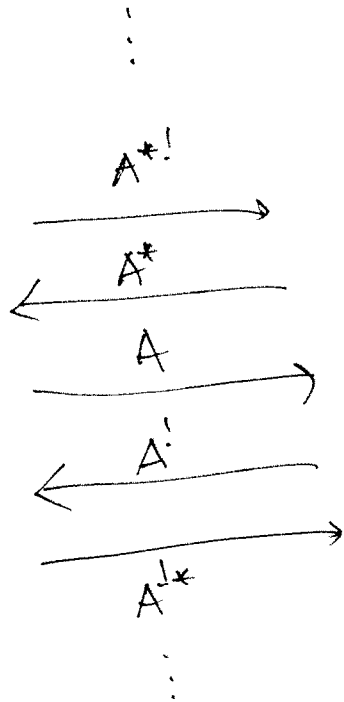
$\text{Hom}_{A^\circ \otimes A}(M, A^\circ \otimes A)$

for M a right $A^\circ \otimes A$ -module

$\Rightarrow \Rightarrow$ left

$A^\circ \otimes A^\circ$ -module.

$k\text{-Mod}$



$A\text{-Mod} \otimes A^\circ\text{-Mod}$