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Around Fully dualizable objects in (∞, n)-categories — an introduction

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Bimod := $\otimes$-bicategory w/ objects: k-algebras (assoc., unital) (k fixed ring)

1-morphisms: bimodules $A \xrightarrow{M} B$, M a left B-module, right A-module

Notation: $A \xrightarrow{{}_B M_A} B$

2-morphisms: maps of bimodules

$\otimes$-structure: $A \otimes B$ is $A \otimes B$ as algebras

$$A \otimes A' \xrightarrow{{}_B M_A \otimes {}_{B'} M'_{A'}} B \otimes B'$$

composition:
$$\begin{tikzcd} A \arrow[r, "{}_B M_A"] \arrow[rr, bend right, "{}_C N_B \otimes_B {}_B M_A"'] & B \arrow[r, "{}_C N_B"] & C \end{tikzcd}$$

CatMod := bicategory w/

objects: algebras A thought of as A-mod = cat. of left A-modules

1-morphisms: $A\text{-mod} \xrightarrow{F} B\text{-mod}$ cocont. functors (commute w/ colimits)

Remarks: ① A-mod and B-mod are presentable categories $\Longrightarrow$ $F : A\text{-mod} \to B\text{-mod}$ is left adjoints iff F is cocontinuous (general adjoint functor theorem)

② All cocontinuous $F : A\text{-mod} \longrightarrow B\text{-mod}$

are of the type $\quad X \longmapsto {}_B M_A \otimes_A X,$

where ${}_B M_A := F(A) \quad A \longmapsto {}_B M_A \otimes_A A = {}_B M_A$

2-morphisms: natural transformations of functors
$\Longleftrightarrow$ maps of bimodules

There is an equivalence of $\underbrace{\text{bicategories}}^{\text{monoidal}}$

$$\text{Bimod} \xrightarrow{\ \sim\ } \text{CatMod}$$

$\otimes$-structure on CatMod

$$A\text{-Mod} \otimes B\text{-Mod} := A \otimes B\text{-mod}$$

Now to every bimodule we associate an adjunction

$$\begin{tikzcd} A\text{-mod} \arrow[r, shift left, "M \otimes_A -"] & B\text{-mod} \arrow[l, shift left, "\mathrm{Hom}_B(M,-)"] \end{tikzcd}$$

$\forall X \in A\text{-mod},\ Y \in B\text{-mod}$

$$\frac{M \otimes_A X \longrightarrow Y \qquad \text{in } B\text{-mod}}{X \longrightarrow \mathrm{Hom}_B(M, Y) \qquad \text{in } A\text{-mod}}$$

(i.e. there is a bijection btw. the set of the maps on top and the set of the maps on the bottom.)

<u>Question:</u> When does a bimodule ${}_B M_A$ have a left or right adjoint in Bimod?

<u>Answer:</u> ${}_B M_A$ has a right adjoint if it is projective of finite type /B

${}_B M_A$ has a left adjoint " " /A

<u>right adjoint:</u>
The question reduces to:

When is $\mathrm{Hom}_B(M,-)$ isomorphic to ${}_A N_B \otimes_B -$?

If such an iso exists, $N = \mathrm{Hom}_B(M,B)$!

Thus, we want to know when

$$\mathrm{Hom}_B(M,Y) \simeq \mathrm{Hom}_B(M,B) \otimes_B Y \quad ?$$

<u>A:</u> When M is projective of finite type/B.

<u>left adjoint:</u>

$$\begin{tikzcd} A\text{-mod} \arrow[r, "M\otimes_A -"] & B\text{-mod} \arrow[l, bend right, "N\otimes_B -"'] \arrow[l, bend left, "\mathrm{Hom}_B(M,B)\otimes_B -"] \end{tikzcd}$$

$\exists$ left adj. if $\exists N$ s.t. $\mathrm{Hom}_A(N,A) \simeq M$.

$N = \mathrm{Hom}_A(M,A)$ is OK if M proj. of f.t./A.

$$
\begin{tikzcd}[column sep=huge]
\vdots \\
\mathrm{A\text{-}mod} \arrow[r, shift left=6, "\mathrm{Hom}_B(\mathrm{Hom}_A(M,A),B)"] & \arrow[l, shift right=3, "\mathrm{Hom}_A(M,A)"'] \mathrm{B\text{-}mod} \\
\vdots
\end{tikzcd}
$$

$$
\mathrm{A\text{-}mod} \xrightarrow{M} \mathrm{B\text{-}mod}, \qquad \mathrm{A\text{-}mod} \xleftarrow[\mathrm{Hom}_B(M,B)]{} \mathrm{B\text{-}mod}, \qquad \mathrm{A\text{-}mod} \xrightarrow[\mathrm{Hom}_A(\mathrm{Hom}_B(M,B))]{} \mathrm{B\text{-}mod}
$$

(if they exist.)

$\vdots$

<u>Question</u>: If M is projective of finite type over A and B, does this imply that all duals are projective of finite type over A and B?

<u>dualizable objects in Bimod?</u>

<u>Recall</u>: The dual of A is A°.

ev: $A^\circ \otimes A \xrightarrow{{}_k A_{A^\circ \otimes A}} k$

coev: $k \xrightarrow[{}_{A\otimes A^\circ} A_k]{} A \otimes A^\circ$

A is a left $A \otimes A^{op}$-module: $(A \otimes A^{op}) \otimes A \longrightarrow A$

$\alpha \otimes \beta \cdot a \longmapsto \alpha a \beta$

<u>ev</u>: $A^\circ\text{-mod} \otimes A\text{-mod} \longrightarrow k\text{-mod}$

$\quad\; \| $

$A^\circ \otimes A\text{-mod} \xrightarrow{A \otimes_{A^\circ \otimes A} -} k\text{-mod}$

$$X \longmapsto A \underset{A^\circ \otimes A}{\otimes} X = X/[A, X]$$

$$= X/_{ax \sim xa}$$

"coinvariant of X"

coev: $\quad k\text{-mod} \longrightarrow A \otimes A^\circ\text{-mod}$

$\qquad\qquad Y \longmapsto A \otimes Y$

1) ev has a right adjoint if A is projective of finite type $/k$
2) " left " $/A^\circ \otimes A$
3) coev " right " $/A^\circ \otimes A^\circ$
4) " left " $/k$

Note: 1) $\Leftrightarrow$ 4) $\Leftrightarrow$ A finite dimension

3) $\Leftrightarrow$ 2) $\Leftrightarrow$ A separable

Question: Is finite dimensional and separable enough to have the full hierarchy of adjoints?

$$
\begin{tikzcd}[column sep=huge]
A^\circ\text{-Mod} \otimes A\text{-Mod}
\arrow[r, shift left=15, phantom, "\vdots"]
\arrow[r, leftarrow, shift left=12, "A^{!*!}"]
\arrow[r, shift left=8, "A^{!*}"]
\arrow[r, leftarrow, shift left=4, "A^{!}"]
\arrow[r, "A"]
\arrow[r, shift right=4, "A^{*}"]
\arrow[r, leftarrow, shift right=8, "A^{*!}"]
\arrow[r, shift right=12, "A^{*!*}"]
\arrow[r, shift right=16, phantom, "\vdots"]
& k\text{-Mod}
\end{tikzcd}
$$

Where $M^! =$

$\mathrm{Hom}_{A^\circ\otimes A}(M, A^\circ\otimes A)$

for M a right

$A^\circ\otimes A$ - module

~~$\Leftrightarrow$~~ $\Rightarrow$ left

$A\otimes A^\circ$ - module.

$$
\begin{tikzcd}[column sep=huge]
k\text{-Mod}
\arrow[r, shift left=15, phantom, "\vdots"]
\arrow[r, shift left=12, "A^{*!}"]
\arrow[r, leftarrow, shift left=8, "A^{*}"]
\arrow[r, "A"]
\arrow[r, leftarrow, shift right=4, "A^{!}"]
\arrow[r, shift right=8, "A^{!*}"']
\arrow[r, shift right=14, phantom, "\vdots"]
& A\text{-Mod} \otimes A^\circ\text{-Mod}
\end{tikzcd}
$$